# ETUDES ON CONVEX POLYHEDRA. 8. NOVEL EVIDENCE OF ANTISYMMETRY

https://doi.org/10.31241/MIEN.2018.14.08

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## Abstract

The earlier introduced concept of real crystal forms is extended by “partly everted convex polyhedra”. It is illustrated with trigonal trapezohedra and dipyramids, physically interpreted and characterized by classic and antisymmetry point groups.

## Synopsis

The concept of real crystal forms is extended, illustrated with trigonal trapezohedra and dipyramids, physically interpreted and characterized by classic and antisymmetry point groups.

## Key words

Real crystal forms, partly everted convex polyhedra, antisymmetry, trigonal trapezohedron, trigonal dipyramid.

## 1. Introduction

As shown by Voytekhovsky & Stepenshchikov (2017), some old crystallographic ideas can be developed, if reinterpreted. Antisymmetry (black-white or magnetic symmetry) is another idea of the kind. It is introduced by Heesch (1930) and developed by Shubnikov (1951) by combining enantiomorphism (*i.e.* mirror symmetry) of a geometric form and dualism of its physical property. For the latter, black gloves with white linings were the best-used example explaining the idea. The right-hand black glove is enantiomorphous to the left-hand black one. But it is antisymmetric to the left-hand white glove, *i.e.* to itself everted. Following the same methodology, we extend the earlier introduced concept of a “real crystal form”. It was defined as “any convex polyhedron bounded, at least, by some of the planes of a given ideal crystal form in a standard orientation with arbitrary distances from the origin of coordinates” (Voytekhovsky, 2002). It was helpful to describe deformed rhombododecahedra of almandine from Mt Makzapakhk, the West Keyvy Ridge, Kola Peninsula (Voytekhovsky & Stepenshchikov, 2004). The crystal forms appeared to belong to some subgroups of the $m\bar{3}m$ symmetry point group (s.p.g.) of an ideal rhombododecahedron. The above effect was interpreted in accordance with the Curie dissymmetry principle as a growth of almandine crystals variously oriented in the gradient field of matter and heat.

## 2. Everting a convex polyhedron

In the above case, the facets of convex polyhedra simulated those of crystals growing with different speeds in different directions. But what happens, if we move the facets in parallel to the opposite side of the origin of coordinates? If we move all facets of any primitive or axial convex polyhedron to the opposite side of the origin of coordinates at the same distances from the latter, we get an everted enantiomorphous polyhedron. Internal normals to the facets of an initial polyhedron (*i.e.* oriented into it) turn to the external normals of a new one (*i.e.* oriented out of it) and *vice versa*. In this sense, they are painted the opposite color. Thus, the resulted polyhedron is antisymmetric to the initial one. If an initial polyhedron isn't of primitive or axial symmetry, then an everted one is of the same symmetry and painted the opposite color (Fig. 1).

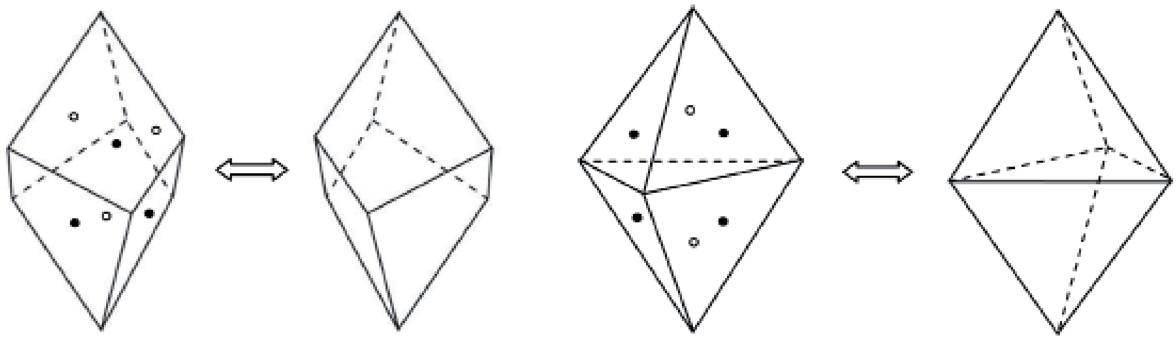

*Figure 1. Antisymmetric trigonal trapezohedra (left) and trigonal dipyramids painted the opposite color (right). Hereinafter the internal normals are marked black (on front facets) and white (on back facets) circles, the external normals are not marked.*

## 3. Results and discussion

For simplicity, let us name a convex polyhedron partly everted, if it results from the initial one by moving some of its facets to the opposite side of the origin of coordinates at any distances from the latter. And let us name it reduced, if some facets are eliminated, *i.e.* moved to infinity, in the above procedure. In combinatorial approximation, *i.e.* with respect to the number and combination of different (3-, 4-, 5-, . . . , n-gonal) facets, only a finite number of partly everted polyhedra can result from any initial polyhedron. The generating computer algorithm is basically explained in (Voytekhovsky, 2002) with an example of "real crystal octahedra".

### 3.1. Trigonal trapezohedron

Fig. 2 shows polyhedra derived from a trigonal trapezohedron. They are characterized by the s.p.g.'s in Table 1. The only polyhedron # 63 is of the ***m'*** antisymmetry point group. The others are of the ***1*** and ***2*** classic s.p.g.'s. Note that the ***m*** s.p.g. is not a subgroup of the ***32*** s.p.g. of a trigonal trapezohedron (Litvin, 2008).

Table 1. S.p.g's of convex polyhedra derived from a trigonal trapezohedron. The numbers correlate with Fig. 2.

| s.p.g.'s | ***1*** | ***2*** | ***32*** | ***m'*** |
|---|---|---|---|---|
| 4-hedra | 1 | 2-4 | | |
| 5-hedra | 5-19 | | | |
| 6-hedra | 20, 24-33, 40-44, 48-59 | 21, 22, 34-39, 45-47, 60-62 | 23 | 63 |

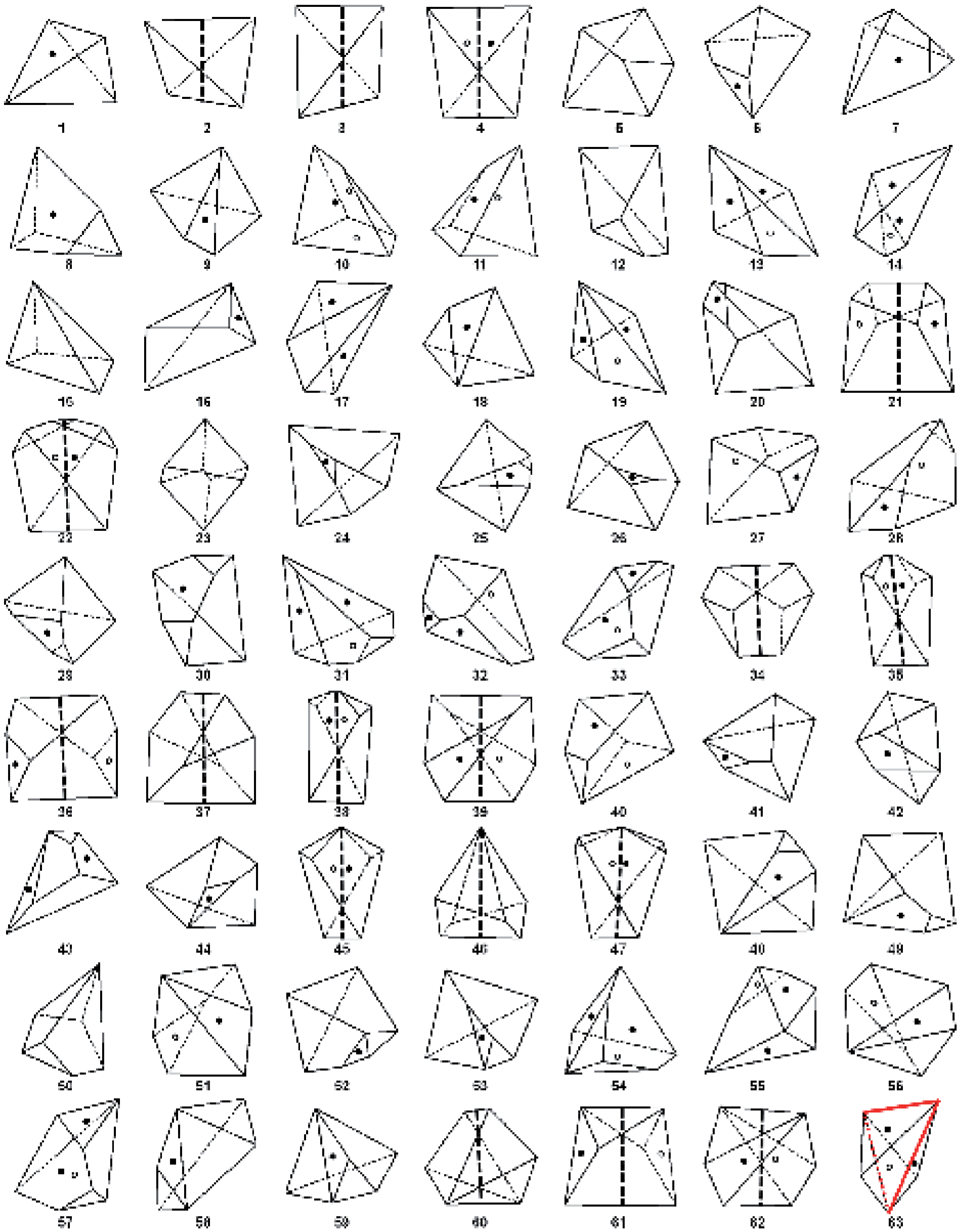

*Figure 2. Convex polyhedra derived from a trigonal trapezohedron (# 23). The polyhedra # 2, 3, 5, 12, 15, 23, 34, 37, 46, 50, and 60 are normal (with external normals only), the others (52 in total) are partly everted (with some internal normals), # 1-19 are reduced (the numbers of facets equal 4 or 5 ). The axes of symmetry (black, except for # 23) and a plane of antisymmetry (red, # 63) are marked.*

### 3.2. Trigonal dipyramid

Fig. 3 shows polyhedra derived from a trigonal dipyramid. They are characterized by the s.p.g.'s in Table 2. The polyhedra # 2, 21 and 35 are of the ***2'2'2***, ***2'*** and ***mm'2'*** antisymmetry point groups, respectively. The others are of the ***1***, ***2*** and ***m*** classic s.p.g.'s. Note that the ***222*** s.p.g. is not a subgroup of the $\bar{6}m2$ s.p.g. of a trigonal dipyramid.

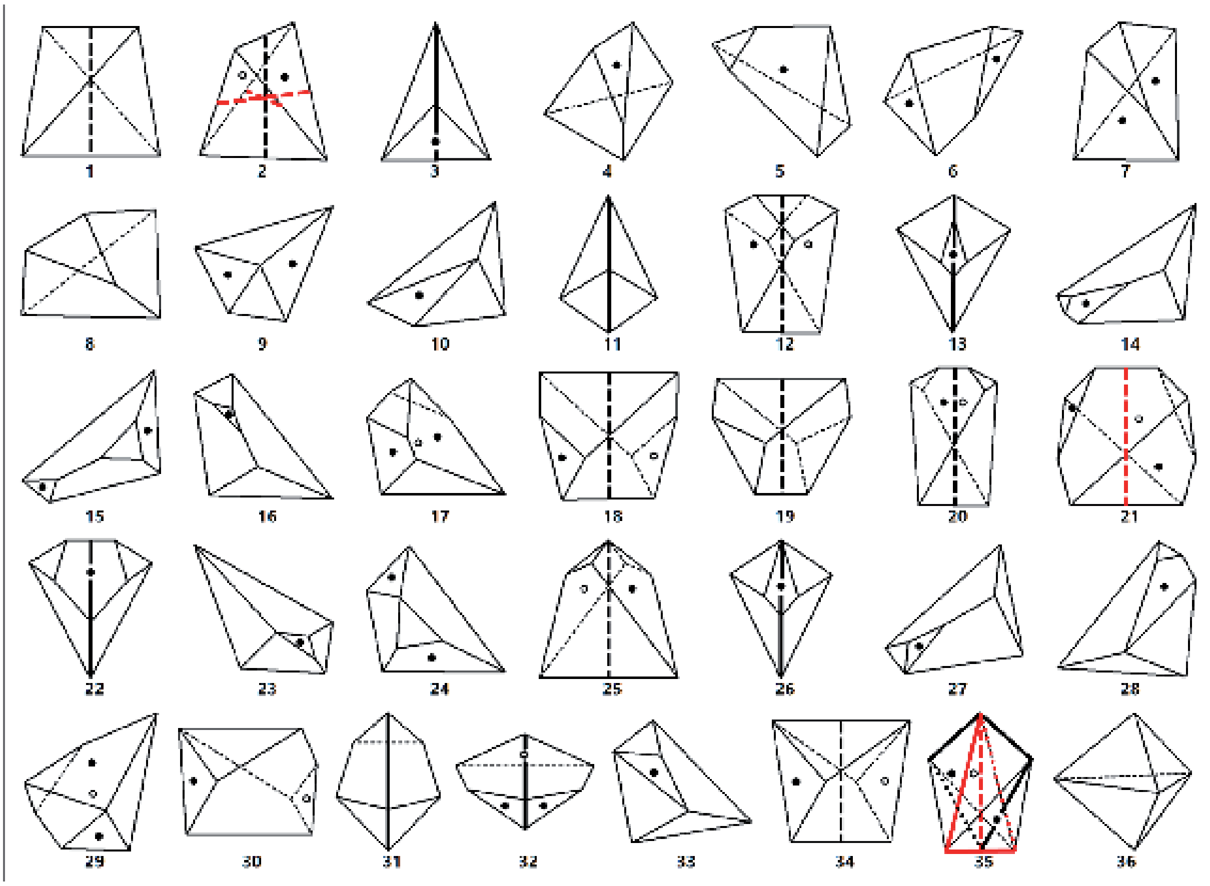

*Figure 3. Convex polyhedra derived from a trigonal dipyramid (# 36). The polyhedra # 1, 8, 11, 19, 31, and 36 are normal, the others (30 in total) are partly everted, # 1-11 are reduced. The axes of symmetry (black, except for # 36), the axes and a plane of antisymmetry (red, # 2, 21 and 35) are marked.*

Table 2. S.p.g's of convex polyhedra derived from a trigonal dipyramid. The numbers correlate with Fig. 3.

| s.p.g.'s | ***1*** | ***2*** | ***m*** | $\bar{6}m2$ | ***2'*** | ***2'2'2*** | ***mm'2'*** |
|---|---|---|---|---|---|---|---|
| 4-hedra | | 1 | 3 | | | 2 | |
| 5-hedra | 4-10 | | 11 | | | | |
| 6-hedra | 14-17, 23, 24, 27-30, 33 | 12, 18-20, 25, 34 | 13, 22, 26, 31, 32 | 36 | 21 | | 35 |

### 3.3. Antisymmetric polyhedra

The most interesting question is how antisymmetric partly everted polyhedra do result in the above procedures with their s.p.g.'s being no subgroups of the s.p.g.'s of initial polyhedra. It is easy to see that all partly everted polyhedra contain some facets of both initial and everted polyhedra. It looks like any partly everted polyhedron results from a combined (initial + everted) antisymmetric polyhedron. The obvious restriction is that no more than a half of its facets are used. Thus, an initial polyhedron (also everted one) is hemihedral, if compared with a combined one. That is why the s.p.g.'s of the partly everted polyhedra are the subgroups of the s.p.g.'s of combined polyhedra despite the fact that they are generated from the initial ones. For example, a trigonal trapezohedron produces a combined ditrigonal scalenohedron of the $\bar{3}'m'$ s.p.g. with ***m***' being its subgroup. In the same way, a trigonal dipyramid produces a combined hexagonal dipyramid of the ***6'/mmm'*** s.p.g. with ***2***', ***2'2'2*** and ***mm'2'*** being its subgroups (Fig. 4).

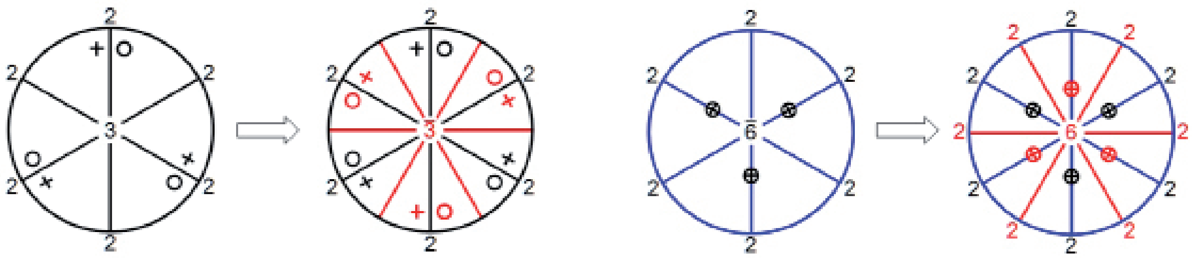

*Figure 4. The initial, everted and combined convex polyhedra in the stereographic projections. A trigonal trapezohedron (s.p.g.* ***32****) generates a ditrigonal scalenohedron (s.p.g.* $\bar{3}'m'$*, left), a trigonal dipyramid (s.p.g.* $\bar{6}m2$*) generates a hexagonal dipyramid (s.p.g.* ***6'/mmm'****, right). The facets are indicated with the crosses (front hemisphere) and circles (back hemisphere), black (initial) and red (everted). The axes of symmetry (****2, 3,*** $\bar{6}$*) and antisymmetry (****2,*** $\bar{3}$*,* ***6****) are marked black and red figures, while the planes of symmetry and antisymmetry are marked blue and red, respectively.*

### 3.4. Physical interpretation

Partly everted convex polyhedra can be physically interpreted. The first idea is to consider the facets with external normals as those of crystal growth and the facets with internal normals as those of crystal dissolution. Crystal twins with some facets of the two enantiomorphous individuals are another interpretation. Both of them seem to be possible, at least, theoretically.

## 4. Conclusions

The paper expands the earlier reported concept of "a real crystal form as any convex polyhedron bounded, at least, by some of the planes of a given ideal crystal form in a standard orientation with arbitrary distances from the origin of coordinates". A convex polyhedron is named partly everted, if it results from the initial one by moving some of its planes to the opposite side of the origin of coordinates at any distances from the latter. A convex polyhedron is named reduced, if some facets are eliminated (moved to infinity) in the above procedure illustrated with trigonal trapezohedra and dipyramids.

It is found that some everted polyhedra are of the antisymmetry point groups, which are not subgroups of the s.p.g.'s of the initial polyhedra. The everted polyhedra are shown to result from the combined (initial + everted) polyhedron: a trigonal trapezohedron generates a ditrigonal scalenohedron of the $\bar{3}'m'$ s.p.g., while a trigonal dipyramid generates a hexagonal dipyramid of the ***6'/mmm'*** s.p.g. Therefore, they are allowed to have the s.p.g.'s being the subgroups of the s.p.g.'s of the composed polyhedra: ***m'*** of $\bar{3}'m'$ and ***2'***, ***2'2'2***, ***mm'2'*** of ***6'/mmm'***.

The "partly everted convex polyhedra" are physically interpreted as crystals with the facets of growth and dissolution or as crystal twins with the facets of both enantiomorphous individuals.

**Acknowledgements**

The authors are grateful to the unknown referees for the highly skilled comments.

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## РАЗВИТИЕ МЕТОДА ОЦЕНКИ ВИДИМОЙ СИММЕТРИИ ИСКАЖЁННОГО КРИСТАЛЛА

https://doi.org/10.31241/MIEN.2018.14.09

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**Анотация**

В статье рассмотрен малоизвестный способ оценки видимой симметрии искажённых кристаллов с помощью отношения площадей граней, переходящих друг в друга при симметрических преобразованиях. Предлагается вниманию дальнейшее развитие метода.