5.2.2. r = n–2

More so for the case r = n–2. Here, two rows of the adjacency matrix cannot be found using its symmetry. But they can be calculated from the others. How to interpret this property? Fig. 4 shows that if four rows of the adjacency matrix (*ex*. # 1 to 4) are fixed, then only the elements (5, 6) and (6, 5) are unfixed. And (5, 6) = (6, 5) = 1 to guarantee the polyhedral property of the edge graph. This interpretation seems proper for any case r = n–2.

6. Conclusion

Suggested is the concept of convex 0-polyhedra, for which the determinants of the adjacency matrices of the edge graphs equal 0. The simplest case, when some rows of the adjacency matrices are identical, is investigated in details. Such 0-polyhedra form two endless series: the “dipyramids” (s.p.g.’s $\overline{(2n-4)}m2$ for odd n, s.p.g.’s (n-2)/mmm for even n; a special case – an octahedron: s.p.g. $m\bar{3}m$, 3 pairs of identical rows), and the “ridge-type” polyhedra (s.p.g. mm2; a special case – a tetragonal pyramid: s.p.g. 4mm, 2 pairs of identical rows). The concept of the rank r of adjacency matrices is useful to describe a general case. In this case, n–r rows of adjacency matrices can be calculated from the others even for the combinatorially asymmetric 0-polyhedra. Thus, linear relations between the rows of adjacency matrices of convex 0-polyhedra are their fundamental property independent from the symmetry.

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ETUDES ON CONVEX POLYHEDRA. 7. THE ROME DE LISLE PROBLEM

https://doi.org/10.31241/MIEN.2018.14.07

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Abstract

The “Rome de Lisle problem” on the vertex and edge truncations has been formulated and solved for all crystal closed simple forms (2, 8, 5, and 15 for orthorhombic, trigonal + hexagonal, tetragonal, and cubic syngonies, respectively). The collections of simple forms obtained are enumerated and considered as special combinations of simple forms in symmetry classes.

Synopsis

The vertex and edge truncations of all crystal closed simple forms have been enumerated and interpreted as special combinations of crystal simple forms.

Key words

Crystal closed simple forms, symmetry classes, vertex and edge truncations, Rome de Lisle.

1. Introduction

A crystal polyhedron (*i.e.* any group of crystal faces that form a convex polyhedron) is currently considered from the viewpoint of relative position of its faces. This tradition is justified by the 1st law of crystallography (law of constancy of angles) and the goniometric technique of measuring. But at the dawn of the science Werner (1786) differentiated crystals according to their vertices, while Rome de Lisle (1772) valued all elements. He noticed that a crystal could be truncated in its vertices or / and along edges and suggested referring them to the same class as the main form. Fedorov (1893) suggested α, β and γ cutting operations in his recurrence algorithm to generate the whole combinatorial variety of convex polyhedra from a tetrahedron. Goldschmidt (1921) dealt with edge truncations in his "complication rule". Voytekhovsky (2016, 2017 *a*, *b*) showed the issue to be conventional up to date, *i.e.* in some cases a convex polyhedron can be reasonably considered from the point of its vertices or edges. For certainty, the authors formulate the following "Rome de Lisle problem": to define a form resulting from the truncation of vertices or edges for the given crystal closed simple form. It is below solved for all crystal closed simple forms as they are in crystallography: 2, 8, 5, and 15 for orthorhombic, trigonal + hexagonal, tetragonal, and cubic syngonies, respectively (Tables 1, 2). The article is dedicated to the crystallography forefathers, who left a number of unsolved problems.

2. Vertex and edge truncations

Equivalent (*i.e.* linked by symmetry operations) vertices and edges are equally truncated: (a) the truncation planes are equally oriented to the faces that meet at vertices and edges; (b) the truncation planes are at same distances from the centre of a polyhedron; (c) the truncations are made up to the midpoints of the original faces, *i.e.* they vanish.

Generally, vertex truncations result in geometrically dual forms. For example, truncating vertices of an octahedron, we get a cube. But truncating vertices of a cube, we get an octahedron ($m\bar{3}$, 432, $m\bar{3}m$ classes) or 2 tetrahedra (23, $\bar{4}3m$). Examples are widespread in nature, *ex*. an octahedron is dual to a cube on crystals of fluorite and diamond, a combination of a prism and pinacoid is dual to the same-name dipyramid on crystals of topaz, apatite, *etc*. Dual forms occur on crystals jointly (*i.e.*, octahedron + cube on a fluorite crystal) and separately (*i.e.*, an octahedron and a cube on different fluorite crystals). These observations reveal the natural background in the "Rome de Lisle problem" and allow rewording the problem of vertex truncations: which combinations of simple forms are dual to the closed simple ones in each class of symmetry?

There is a rigid ratio between original closed simple forms and polyhedra resulted from truncations, considering a number of faces (f, F), vertices (v, V) and edges (e, E). In both cases they correspond to the Euler ratio: $f + v = e + 2$, $F + V = E + 2$. For vertex truncations we have $F = v$, $V = f$, hence:

$$E = F + V - 2 = v + f - 2 = e.$$

Similarly, for edge truncations we have $F = e$, $V = f + v$, hence:

$$E = F + V - 2 = e + (f + v) - 2 = e + (e + 2) - 2 = 2e.$$

Tables 1 and 2 provide the numbers of vertex orbits or edge orbits, respectively, for the original closed simple forms (bracketed). They correspond with the numbers of equivalent faces in obtained collections of simple forms. The authors have computerized the procedure of truncation of vertices and edges. The results are represented in Tables 1, 2 and Fig. 1. The above ratios allow identifying crystal simple forms in combinations.

Table 1. Vertex truncations of closed simple forms (c.s.f.).

	Initial c.s.f.	**Vertex truncations**
	Orthorhombic syngony	
1	rhombic disphenoid (4)	rhombic disphenoid
2	rhombic dipyramid (2+2+2)	3 pinacoids
	Trigonal and hexagonal syngony	
3	trigonal dipyramid (3+2)	trigonal prism + pinacoid
4	rhombohedron (6+2)	rhombohedron + pinacoid
5	trigonal trapezohedron (6+2)	trigonal trapezohedron + pinacoid
6	ditrigonal dipyramid (6+2)	ditrigonal prism + pinacoid
7	ditrigonal scalenohedron (6+2)	rhombohedron + pinacoid
8	hexagonal trapezohedron (12+2)	hexagonal trapezohedron + pinacoid
9	hexagonal dipyramid (6+2)	hexagonal prism + pinacoid
10	dihexagonal dipyramid (12+2)	dihexagonal prism + pinacoid
	Tetragonal syngony	
11	tetragonal disphenoid (4)	tetragonal disphenoid
12	tetragonal dipyramid (4+2)	tetragonal prism + pinacoid
13	tetragonal scalenohedron (4+2)	tetragonal disphenoid + pinacoid
14	tetragonal trapezohedron (8+2)	tetragonal trapezohedron + pinacoid
15	ditetragonal dipyramid (8+2)	ditetragonal prism + pinacoid
	Cubic syngony	
16	tetrahedron (4)	tetrahedron
17	octahedron (6)	cube
18	cube (8)	octahedron ($m\bar{3}$, 432, $m\bar{3}m$) or 2 tetrahedra (23, $4\,3m$)
19	rhombic dodecahedron (8+6)	octahedron ($m\bar{3}$, 432, $m\bar{3}m$) or 2 tetrahedra (23, $4\,3m$) + cube
20	pentagonal dodecahedron (12+8)	pentagonal dodecahedron + octahedron ($m\bar{3}$) or 2 tetrahedra (23)
21	trigonal tristetrahedron (4+4)	2 tetrahedra
22	trapezohedral tristetrahedron (6+4+4)	cube + 2 tetrahedra
23	tetartoid (12+4+4)	tetartoid + 2 tetrahedra
24	hextetrahedron (6+4+4)	cube + 2 tetrahedra
25	trigonal trisoctahedron (8+6)	octahedron + cube
26	trapezohedral trisoctahedron (12+8+6)	rhombic dodecahedron + octahedron + cube
27	gyroid (24+8+6)	gyroid + octahedron + cube
28	tetrahexahedron (8+6)	octahedron (432, $m\bar{3}m$) or 2 tetrahedra ($4\,3m$) + cube
29	diploid (12+8+6)	pentagonal dodecahedron + octahedron + cube
30	hexoctahedron (12+8+6)	rhombic dodecahedron + octahedron + cube

Table 2. Edge truncations of closed simple forms (c.s.f.).

	Initial c.s.f.	Edge truncations
		Orthorhombic syngony
1	rhombic disphenoid (2+2+2)	3 pinacoids
2	rhombic dipyramid (4+4+4)	3 rhombic prisms
		Trigonal and hexagonal syngony
3	trigonal dipyramid (6+3)	trigonal dipyramid + trigonal prism
4	rhombohedron (6+6)	rhombohedron + hexagonal prism
5	trigonal trapezohedron (6+3+3)	trigonal trapezohedron + 2 trigonal prisms
6	ditrigonal dipyramid (6+6+6)	2 trigonal dipyramids + ditrigonal prism
7	ditrigonal scalenohedron (6+6+6)	2 rhombohedra + hexagonal prism
8	hexagonal trapezohedron (12+6+6)	hexagonal trapezohedron + 2 hexagonal prisms
9	hexagonal dipyramid (12+6)	hexagonal dipyramid + hexagonal prism
10	dihexagonal dipyramid (12+12+12)	2 hexagonal dipyramids + dihexagonal prism
		Tetragonal syngony
11	tetragonal disphenoid (4+2)	tetragonal prism + pinacoid
12	tetragonal dipyramid (8+4)	tetragonal dipyramid + tetragonal prism
13	tetragonal scalenohedron (4+4+4)	tetragonal prism + 2 tetragonal disphenoids
14	tetragonal trapezohedron (8+4+4)	tetragonal trapezohedron + 2 tetragonal prisms
15	ditetragonal dipyramid (8+8+8)	ditetragonal prism + 2 tetragonal dipyramids
		Cubic syngony
16	tetrahedron (6)	cube
17	octahedron (12)	rhombic dodecahedron
18	cube (12)	rhombic dodecahedron
19	rhombic dodecahedron (24)	trapezohedral trisoctahedron
20	pentagonal dodecahedron (24+6)	diploid + cube
21	trigonal tristetrahedron (12+6)	trapezohedral tristetrahedron + cube
22	trapezohedral tristetrahedron (12+12)	2 trigonal tristetrahedra
23	tetartoid (12+12+6)	2 tetartoids + cube
24	hextetrahedron (12+12+12)	2 trigonal tristetrahedra + trapezohedral tristetrahedron
25	trigonal trisoctahedron (24+12)	trapezohedral trisoctahedron + rhombic dodecahedron
26	trapezohedral trisoctahedron (24+24)	trigonal trisoctahedron + tetrahexahedron
27	gyroid (24+24+12)	2 gyroids + rhombic dodecahedron
28	tetrahexahedron (24+12)	trapezohedral trisoctahedron + rhombic dodecahedron
29	diploid (24+12+12)	diploid + 2 pentagonal dodecahedra
30	hexoctahedron (24+24+24)	trigonal trisoctahedron + trapezohedral trisoctahedron + tetrahexahedron

3. Discussion

Analysing Tables 1, 2 and Fig. 1 has revealed the following. Vertex and edge truncations have been obtained for each of 30 closed simple forms in its symmetry class. The respective combinations of simple forms are suggested to be treated as special. The result appears as non-trivial, since the geometric crystal morphology allows any combinations of simple forms permissible in the given symmetry class. (In nature constraints are physically predetermined).

The following results have been found for vertex truncations. In a trigonal syngony a rhombohedron and ditrigonal scalenohedron provide different combinations of a rhombohedron and pinacoid. In the first case it looks like a trigonal antiprism, in the second case it is a trigonal antiprism that is cut in parallel to a pinacoid in such a way that trigonal faces have become trapezia. In a cubic syngony a rhombic dodecahedron, trapezohedral tristetrahedron, hextetrahedron, trigonal trisoctahedron and tetrahexahedron provide different combinations of a cube and an octahedron (or two tetrahedra as an example of a hemiedry). A trapezohedral trisoctahedron and hexoctahedron provide different combinations of a rhombic dodecahedron, an octahedron and a cube.

As for the edge truncations, an octahedron and a cube are found to produce a rhombic dodecahedron, while trigonal trisoctahedron and tetrahexahedron

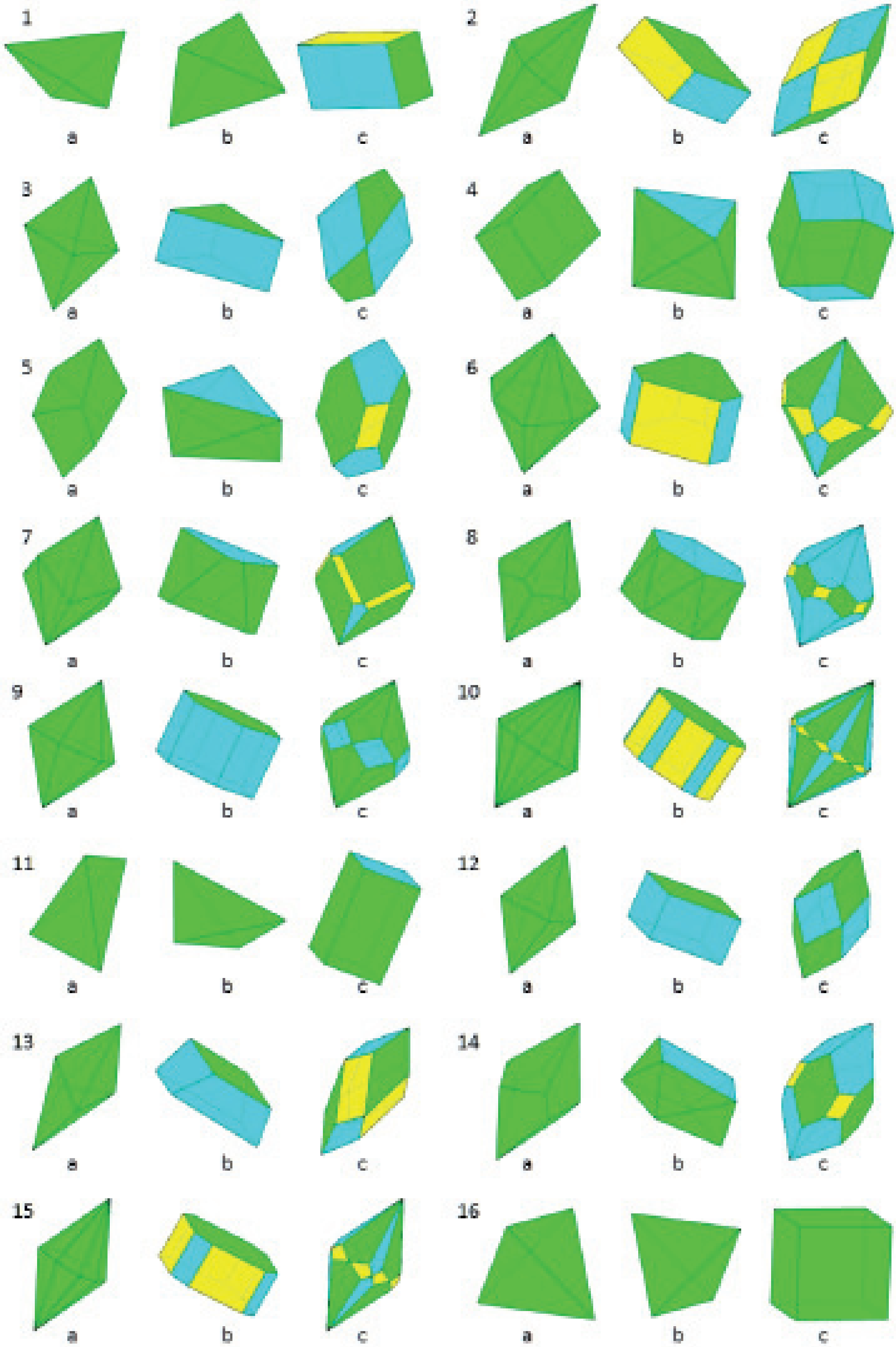

Figure 1. Truncations of closed simple forms. The numbers 1 to 30 fit Tables 1 and 2, a – initial c.s.f., b – vertex truncation, c – edge truncation, different simple forms are marked in the combinations by different colours.

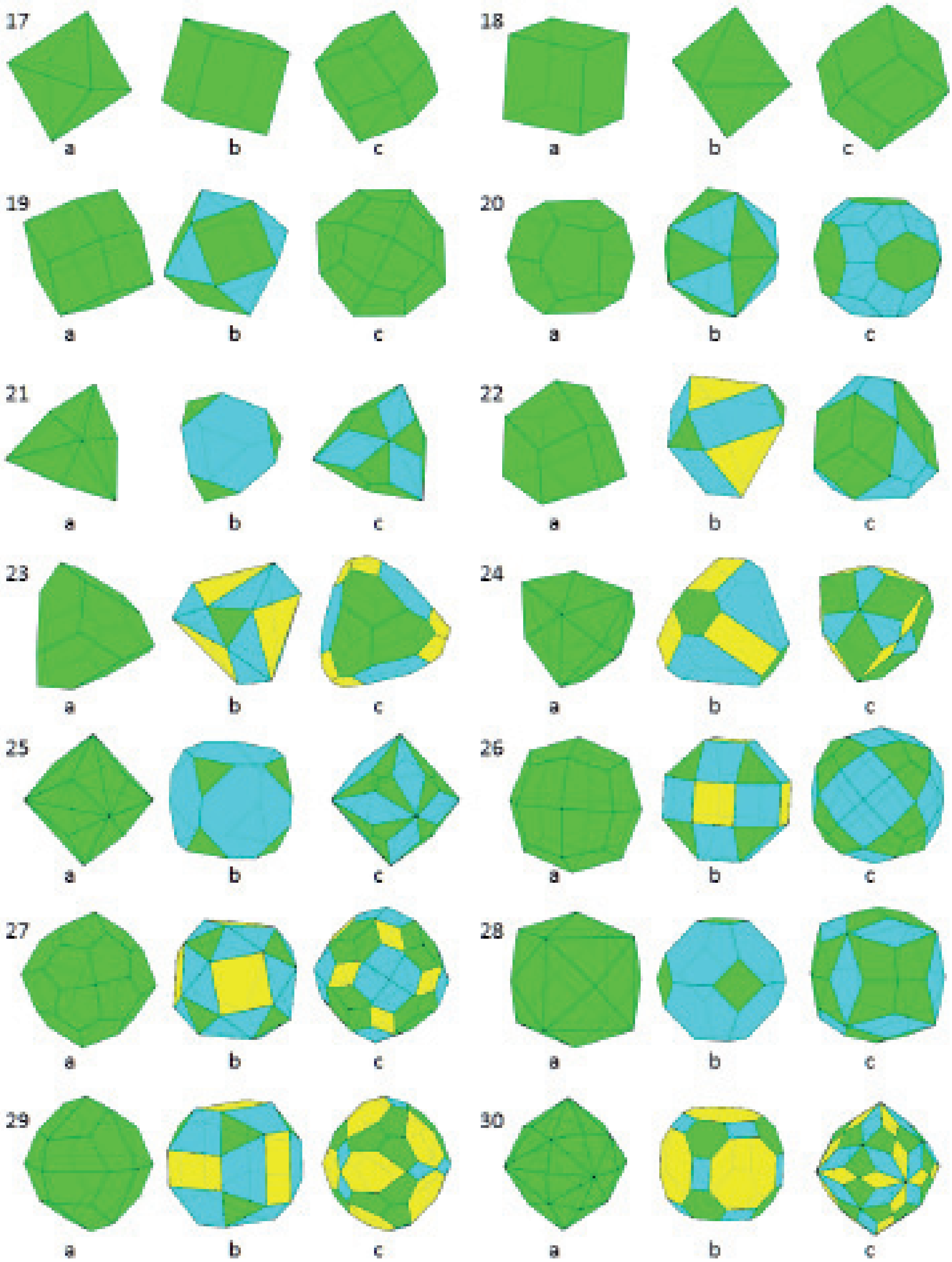

Figure 1. (continued).

produce various combinations of a trapezohedral trisoctahedron and rhombic dodecahedron (in the $\bar{4}3m$ class a trapezohedral trisoctahedron is replaced by two trigonal tristetrahedra – another example of a hemiedry). Various combinations of one and the same simple forms transform one in another by parallel movements of faces along normals. They comply with rotations of faces of original closed simple forms on edges. Thus, faces of a ditrigonal scalenohedron, merging in pairs in a parallel position, produce faces of a rhombohedron. It indicates the affinity of some closed simple forms that differs from the well-known holo-, hemi-, tetarto- and ogdoedrie.

4. Conclusions

Works of forefathers in crystallography are rich in observations and ideas that can be studied in terms of the contemporary science. The “Rome de Lisle problem”, as it is stated above, has been solved for the 30 closed simple forms. Their vertex and edge truncations have been identified as combinations of simple forms in respective symmetry classes.

The affinity of some closed simple forms differing from the well-known holo-, hemi-, tetarto- and ogdoedrie has been defined. Their vertex and / or edge truncations are different combinations of the same simple forms turning one into another, when faces are moved in parallel along normals. Yet original closed simple forms transform one in another in result of rotating faces on edges.

Vertex truncations of closed simple forms produce (among others) the dual forms, well-known on natural crystals to follow: an octahedron *vs*. a cube; a dipyramid *vs*. a same-name prism and a pinacoid, *etc*. Thus, the “Rome de Lisle problem” is valued both theoretically and practically.

Acknowledgements

The authors are grateful to the unknown referee for the highly skilled comments that made the article more rigorous.

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