The entropy H_S characterizes a 'disorder' rather than a 'complexity' of convex n-acra. The first one is quite well characterized by s.p.g's. The second one should distinguish n-acra of the same s.p.g. and different numbers of edges, for example, the overwhelming majority of combinatorially asymmetric n-acra for given $n \geq 7$. To do this, the topological entropy H_V is suggested, which considers the valences of vertices of n-acra. It classifies the variety of convex 4- to 9-acra in more details. It is proved that H_V can reach 0 as minimum (for example, for regular and semi-regular polyhedra, as well as the infinite series of prisms and antiprisms), but never lg n as maximum, because there are no convex n-acra with all vertices of different valences. It is also proved that $H_S \geq H_V$ for any convex n-acron, *i.e*. for any n and s.p.g. $H_S = H_V$ if the vertices non-equivalent under the automorphism group also have different valences, and $H_S > H_V$ if not.

Acknowledgements

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ETUDES ON CONVEX POLYHEDRA. 6. CONVEX 0-POLYHEDRA

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Abstract

New tools to describe a convex polyhedron and their links to the recent theory of crystal morphology are discussed in the paper. Zero-polyhedra, *i.e*. those with a 0 determinant of adjacency matrices of edge graphs, are found to prevail among convex

4- to 7-vertex polyhedra and have strong relationships between their edges. The case when some rows of adjacency matrices are identical is investigated in details with combinatorial types and symmetry point groups of related polyhedra found.

Synopsis

A special class of convex 0-polyhedra is defined, where determinants of adjacency matrices of edge graphs equal 0, with their combinatorial types and symmetry point groups found.

Key words

Convex polyhedra and polyacra, adjacency matrix, determinant, symmetry point group, automorphism group order.

1. Introduction

Crystal polyhedra are traditionally described in terms of symmetry. All combinatorial types of convex 4- to 12-hedra and simple (only 3 facets / edges meet at each vertex) 13- to 16-hedra have been enumerated and characterized by automorphism group orders (a.g.o.'s) and symmetry point groups (s.p.g.'s) in the papers (Voytekhovsky & Stepenshchikov, 2006; Voytekhovsky, 2014). It is found that asymptotically (with growing n) almost all convex n-hedra (and n-acra, *i.e.* n-vertex polyhedra, because of their duality) are combinatorially asymmetric (*i.e.* primitive triclinic). Hence, a problem arises: how to operate them if s.p.g.'s do not work? In the series of papers we investigate new tools to describe a convex polyhedron and their links to the recent theory of crystal morphology.

A general theory of convex polyhedra is given in Grünbaum (1967). Voytekhovsky (2016) has suggested a method of naming any convex n-acron by a numerical code arising from the adjacency matrix of its edge graph. The number of names for any convex n-acron equals n! / a.g.o. depending on labeling of vertices. They are connected with each other by permutations of the same-name rows and columns of the adjacency matrix. Such transformations do not change its determinant Δ, thus, being an important characteristic of a related polyhedron. Here we study combinatorial types and s.p.g.'s of convex polyhedra in a special case Δ = 0.

2. Determinants of convex 4- to 7-acra

The Δ values have been calculated for all convex 4- to 7-acra (Table 1). Zero-polyhedra, *i.e.* those with Δ = 0, prevail among 5- to 7-acra (2 of 2, 4 of 7, and 11 of 34, respectively). They have been extracted from the paper (Voytekhovsky, 2016) and are shown in Fig. 1 in the Schlegel projections on a facet.

Table 1. Numbers of convex n-acra with different Δ values.

n	Δ																		
	-6	-5	-4	-3	-2	-1	**0**	1	2	3	4	5	6	7	8	9	10	11	12
4				1															
5							**2**												
6		1				1	**4**			1									
7	1		2		1		**11**		9		2		3		3		1		1

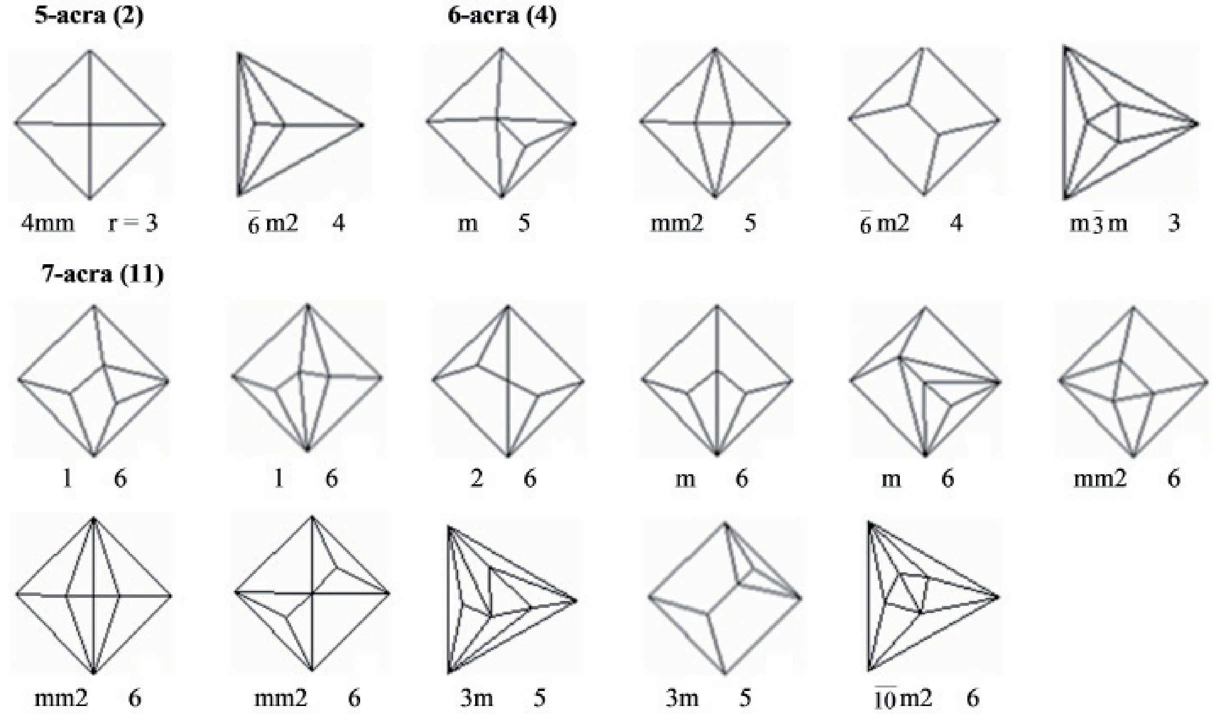
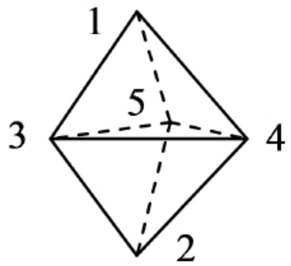


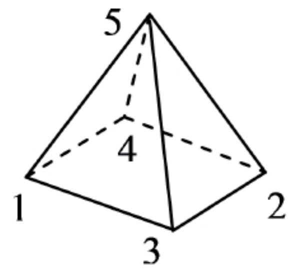
Figure 1. 5- to 7-acra with Δ = 0. S.p.g.'s and ranks (r) of adjacency matrices are given.

3. Why 0?

Fig. 2 shows the two 5-acra with their adjacency matrices. It is easy to see that the 1st and the 2nd rows (and columns, hereinafter) of the matrix are identical for the trigonal dipyramid, because its vertices # 1 and # 2 are equally adjacent to the others. Even two pairs of the rows (1st and 2nd, 3rd and 4th) are identical for the tetragonal pyramid for the same reason. And it follows from the general properties of the determinants that Δ = 0 for such matrices. The above said can be generalized in the following statement.



$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure 2. A trigonal dipyramid and a tetragonal pyramid with their adjacency matrices related to the labeling of the vertices.

Statement. Convex 0-polyhedra with identical rows of adjacency matrices form two endless series of "dipyramids" and "ridge-type" polyhedra. Only two 0's can be in identical rows, *i.e.* only two rows can be identical, *i.e.* only two vertices can be equally adjacent to the others. But, two (a tetragonal pyramid) and three (an octahedron) pairs of the identical rows are allowed.

Proof. Let the i-th and j-th rows of an adjacency matrix be identical. As its diagonal elements (i, i) = (j, j) = 0 then (i, j) = (j, i) = 0. That is, the i-th and j-th vertices of the n-acron are not adjacent. Consider, only two 0's are in the i-th and j-th rows. That is, the i-th and j-th vertices are adjacent to all other. Fig. 3 shows the appropriate graph (centre). Let us transform it to a convex polyhedron by adding edges.

It is easy to see that we can build polyhedra of two types: (a) the "dipyramids", if only triangular facets are allowed, and (b) the "ridge-type" polyhedra, if a quadrilateral facet is allowed. Only one quadrilateral facet is allowed in the latter case. Otherwise, two such facets form an edge (i, j). But, the i-th and j-th vertices are not adjacent. The above polyhedra (considered as n-acra) have the following s.p.g.'s: (a) $\overline{(2n{-}4)}m2$ (odd $n \geq 5$), $m\bar{3}m$ ($n = 6$, an octahedron), (n-2)/mmm (even $n \geq 8$); (b) 4mm ($n = 5$, a tetragonal pyramid), mm2 ($n \geq 6$).

Actually, only two 0's can be in the identical i-th and j-th rows of the adjacency matrix. To prove it, let us try to add new vertices to the Schlegel projections (Fig. 3 a, b). They are to be located inside the triangular (a, b) or quadrilateral (b) facets and adjacent to all vertices except the i-th and j-th. In the 1st case they are adjacent to two nearby vertices of a triangular facet. Besides, in the 2nd case they can be adjacent to two opposite vertices of a quadrilateral facet. In both cases the resulted graph is planar and 2-connected, *i.e.* not polyhedral (which is planar and 3-connected).

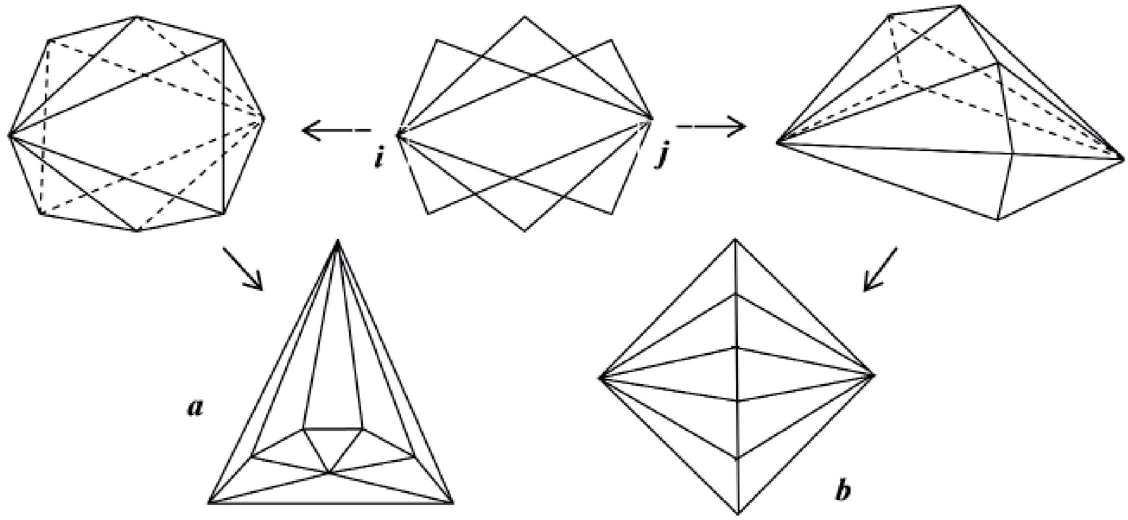
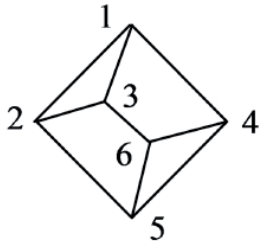


Figure 3. Two types of 0-polyhedra with identical rows of adjacency matrices in the axonometric and Schlegel projections: (a) the "dipyramids", (b) the "ridge-type" polyhedra.

4. A general case

In a general case $\Delta = 0$, but there are no identical rows of the adjacency matrices. To characterize such polyhedra we use the concept of the matrix rank r, *i.e.* the maximum number of its linearly independent rows. Fig. 1 shows ranks of adjacency matrices of 5- to 7-vertex 0-polyhedra. Obviously, $r < n$ for any n-vertex 0-polyhedron. In our case, $n{-}2 \leq r < n$ with the only exception of an octahedron (r = n–3 = 3). Let us consider a trigonal prism (Fig. 4). Its four rows are independent. Hence, any two (n–r = 6–4 = 2) rows depend on the others. For example, the rows # 1, 2, 3, and 4 can be taken as independent. Then the rows # 5 and 6 can be calculated.



$\bar{6}m2$ $\quad r = 4$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$(5) = (1) + (4) - (2)$$

$$(6) = (1) + (4) - (3)$$

Figure 4. 0-polyhedron with linearly dependent rows.

5. Discussion

Convex 0-polyhedra of two types are distinguished, namely, with identical rows of adjacency matrices and without them.

5.1. Identical rows

These form two endless series of convex 0-polyhedra: the “dipyramids” (s.p.g.’s $\overline{(2n-4)}m2$ for odd n, s.p.g.’s (n-2)/mmm for even n; a special case – an octahedron: s.p.g. $m\bar{3}m$, 3 pairs of identical rows), and the “ridge-type” polyhedra (s.p.g. mm2; a special case – a tetragonal pyramid: s.p.g. 4mm, 2 pairs of identical rows). All of them possess rather high symmetries. It looks naturally because in any case pairs of identical rows of the adjacency matrices relate to the symmetry plains of polyhedra.

5.2. No identical rows

The concept of the rank r of adjacency matrices is useful to describe such 0-polyhedra. Obviously, r < n, *i.e.* only r rows of the adjacency matrices are independent, while n–r rows can be calculated. Note that two combinatorially asymmetric 0-polyhedra (s.p.g. 1, r = 6) are among 7-acra (Fig. 1). That is, linear dependence of the rows of adjacency matrices does not mean the polyhedra symmetry. The hypothesis looks plausible that r = n–1 or r = n–2 for any 0-polyhedron with the only exception of an octahedron: r = n–3, where 3 relates to the dimension of the Euclidean space.

5.2.1. r = n–1

In this case any row of the adjacency matrix can be calculated from the others. At the same time, it can be easily found using its symmetry. The case looks trivial. But it is not, because many convex polyhedra have Δ ≠ 0 and, at the same time, symmetrical adjacency matrices.

5.2.2. r = n–2

More so for the case r = n–2. Here, two rows of the adjacency matrix cannot be found using its symmetry. But they can be calculated from the others. How to interpret this property? Fig. 4 shows that if four rows of the adjacency matrix (*ex*. # 1 to 4) are fixed, then only the elements (5, 6) and (6, 5) are unfixed. And (5, 6) = (6, 5) = 1 to guarantee the polyhedral property of the edge graph. This interpretation seems proper for any case r = n–2.

6. Conclusion

Suggested is the concept of convex 0-polyhedra, for which the determinants of the adjacency matrices of the edge graphs equal 0. The simplest case, when some rows of the adjacency matrices are identical, is investigated in details. Such 0-polyhedra form two endless series: the "dipyramids" (s.p.g.'s $\overline{(2n{-}4)}m2$ for odd n, s.p.g.'s (n-2)/mmm for even n; a special case – an octahedron: s.p.g. $m\bar{3}m$, 3 pairs of identical rows), and the "ridge-type" polyhedra (s.p.g. mm2; a special case – a tetragonal pyramid: s.p.g. 4mm, 2 pairs of identical rows). The concept of the rank r of adjacency matrices is useful to describe a general case. In this case, n–r rows of adjacency matrices can be calculated from the others even for the combinatorially asymmetric 0-polyhedra. Thus, linear relations between the rows of adjacency matrices of convex 0-polyhedra are their fundamental property independent from the symmetry.

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ETUDES ON CONVEX POLYHEDRA. 7. THE ROME DE LISLE PROBLEM

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Abstract

The "Rome de Lisle problem" on the vertex and edge truncations has been formulated and solved for all crystal closed simple forms (2, 8, 5, and 15 for orthorhombic, trigonal + hexagonal, tetragonal, and cubic syngonies, respectively). The collections of simple forms obtained are enumerated and considered as special combinations of simple forms in symmetry classes.