matrices of 8- to 11-acra corresponding to their min. names are in Fig. 4. (Note that they differ in one 1 for odd and even n).

4. Conclusions

The min. names are attainable for pyramidal convex n-acra with the following combinatorial s.p.g.: -43*m* for n = 4 (tetrahedron), (n-1)*mm* for odd n > 4 and (n-1)*m* for even n > 4. The max. names are attained for convex n-acra of a «glued tetrahedrons» type with the following combinatorial s.p.g.: -43*m* for n = 4 (tetrahedron), -6*m*2 for n = 5 (trigonal bipyramid) and *mm*2 for n > 5.

The above results allow us to directly calculate ranges of names [min_n, max_n] for any n without generating the whole combinatorial variety of convex n-acra (*ex*., by the routine recurrence Fedorov algorithm) and calculating names for all of them. The ranges of names for n = 4 to 12 are as follows: [63, 63], [507, 1022], [7915, 32754], [241483, 2096914], [15062603, 268427538], [1902830667, 68718960914], [484034528331, 35184305512722], [247052243600459, 36028779906736402], [252590061511541835, 73786967515992695058].

All the names of the ranges [max_n + 1, min_{n+1} – 1] (*ex*., [64, 506], [1023, 7914], [32755, 241482], *etc*.) obviously correspond to the adjacency matrices of non-polyhedral graphs. This sufficient but not necessary criterion seems to be new.

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References

1. Ctrl+ZGrünbaum, B. (1967). *Convex Polytopes*. New York: Springer.
2. Voytekhovsky, Y. L. (2014). *J. Struct. Chemistry*. 55, 7, 1293-1307.
3. Voytekhovsky, Y. L. (2016). *Acta Cryst*, A72, 582-585.
4. Voytekhovsky, Y. L. (2017). *Acta Cryst*, A73, 77-80.
5. Voytekhovsky, Y. L. & Stepenshchikov, D. G. (2006). *Acta Cryst*, A62, 230-232.

ETUDES ON CONVEX POLYHEDRA. 4. ACCELERATED SCATTERING OF CONVEX POLYHEDRA

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Abstract

The formulas for the minimum (min_n) and maximum (max_n) names in the classes of convex n-acra (*i.e*. n-vertex polyhedra) are found for any n. The asymptotic behavior (as $n \to \infty$) for max_{n+1}/max_n, min_{n+1}/min_n, min_{n+1}/max_n, and max_n/min_n is established. They characterize in detail the accelerated scattering of [min_n, max_n] ranges on a real line.

Synopsis

The formulas for min_n and max_n names in the classes of convex n-acra, as well as asymptotic relationships (as $n \to \infty$) between them, are found. These explain the distribution of $[min_n, max_n]$ ranges on the real line.

Key words

Convex polyhedra and polyacra, minimum and maximum names, asymptotic relationships.

1. Introduction

A general theory of convex polyhedra is given in (Grünbaum, 1967). In a series of papers we considered a special problem on the combinatorial variety of convex n-hedra rapidly growing with n. A method of naming any convex n-acron by a numerical code arising from the adjacency matrix of its edge graph has been suggested by Voytekhovsky (2016). It has also been proved that the $[min_n, max_n]$ ranges of names for the classes of convex n-acra are strictly (without overlapping) ordered. The combinatorial types of convex n-acra with the min_n and max_n names (of pyramidal and 'glued tetrahedra' types, respectively) have been found for any n by Voytekhovsky (2017). In this paper, the latter are calculated from the adjacency matrices of their edge graphs. Afterwards, some asymptotic (as $n \to \infty$) relationships between the min_n and max_n names are found. They explain in detail the distribution of $[min_n, max_n]$ ranges on a real line.

2. Maximum and minimum names of convex n-acra

$$\begin{pmatrix} 1&1&1&1&1&1&1&1&1&1\\ &1&1&1&1&1&1&1&1&1\\ &&1&1&0&0&0&0&0&0\\ &&&0&1&0&0&0&0&0\\ &&&&0&1&0&0&0&0\\ &&&&&0&1&0&0&0\\ &&&&&&0&1&0&0\\ &&&&&&&0&1&0\\ &&&&&&&&0&1\\ &&&&&&&&&0 \end{pmatrix} = \underset{A_{11}}{\begin{pmatrix} 1&1&1&1&1&1&1&1&1&1\\ &1&1&1&1&1&1&1&1&1\\ &&1&1&0&0&0&0&0&0\\ &&&0&0&0&0&0&0&0\\ &&&&0&0&0&0&0&0\\ &&&&&0&0&0&0&0\\ &&&&&&0&0&0&0\\ &&&&&&&0&0&0\\ &&&&&&&&0&0\\ &&&&&&&&&0 \end{pmatrix}} + \underset{B_{11}}{\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ &0&0&0&0&0&0&0&0&0\\ &&0&0&0&0&0&0&0&0\\ &&&0&1&0&0&0&0&0\\ &&&&0&1&0&0&0&0\\ &&&&&0&1&0&0&0\\ &&&&&&0&1&0&0\\ &&&&&&&0&1&0\\ &&&&&&&&0&1\\ &&&&&&&&&0 \end{pmatrix}}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&1&1&1\\ &0&0&0&0&0&1&0&1&1\\ &&0&0&0&1&0&1&0&1\\ &&&0&1&0&1&0&0&1\\ &&&&1&1&0&0&0&1\\ &&&&&0&0&0&0&1\\ &&&&&&0&0&0&1\\ &&&&&&&0&0&1\\ &&&&&&&&0&1\\ &&&&&&&&&1 \end{pmatrix} = \underset{C_{11}}{\begin{pmatrix} 0&0&0&0&0&0&0&1&1&1\\ &0&0&0&0&0&1&0&1&1\\ &&0&0&0&0&0&0&0&0\\ &&&0&0&0&0&0&0&0\\ &&&&0&0&0&0&0&0\\ &&&&&0&0&0&0&0\\ &&&&&&0&0&0&0\\ &&&&&&&0&0&0\\ &&&&&&&&0&0\\ &&&&&&&&&0 \end{pmatrix}} + \underset{D_{11}}{\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ &0&0&0&0&0&0&0&0&0\\ &&0&0&0&1&0&1&0&1\\ &&&0&1&0&1&0&0&1\\ &&&&1&1&0&0&0&1\\ &&&&&0&0&0&0&1\\ &&&&&&0&0&0&1\\ &&&&&&&0&0&1\\ &&&&&&&&0&1\\ &&&&&&&&&1 \end{pmatrix}}$$

Figure 1. Decomposition of the adjacency matrices (upper triangles are shown) corresponding to max. (top) and min. (bottom) names of 11-acra.

The adjacency matrices of n-acra with $\min_n$ and $\max_n$ names have been found by Voytekhovsky (2017, Figs 1, 3). Here, to find the names in an explicit algebraical form, we decompose the matrices into A, B, and C, D blocks (Fig. 1).

Exactly, in a general case, matrix A consists of the first three rows of the original adjacency matrix with 0's in other positions while B is the original adjacency matrix minus A. In the same way, matrix C consists of the first two rows of the original adjacency matrix with 0's in other positions while D is the original adjacency matrix minus C. Such decompositions (after many variants were checked) allowed us to find the $\min_n$ and $\max_n$ names of n-acra in an explicit form and, moreover, to prove some relationships between them.

2.1. Formula for the maximum names

For rather big n (for which the adjacency matrices of the n-acra correspond to Fig. 1), $\max_n = A_n + B_n$, where A_n and B_n can be found by using the properties of arithmetic and geometric progressions:

$$A_n = 10 \uparrow \{[1 + 2 + \ldots + (n\text{-}3)] - 2\} \times (1 + 10^1 + \ldots + 10^{2n-2}) =$$

$$= 10 \uparrow [(n^2\text{-}5n+2)/2] \times (10^{2n-1} - 1) / (10 - 1),$$

$$B_n = 10^1 + 10^4 + \ldots + 10 \uparrow \{[1 + 2 + \ldots + (n\text{-}4)] - 2\} =$$

$$= 10^1 + 10^4 + \ldots + 10 \uparrow [(n^2\text{-}7n+8)/2].$$

It is easier to find the bounds for B_n than its explicit algebraic form. In the decimal form, replacing 10 by 2:

$$\max_n = 2 \uparrow [(n^2\text{-}5n+2)/2] \times (2^{2n-1} - 1) + B_n \text{, where}$$

$$\underset{\cdot}{\mathbf{B}}_n = 2 \uparrow [(n^2\text{-}7n+8)/2] < B_n < 2 \uparrow [(n^2\text{-}7n+10)/2] = \dot{\mathbf{B}}_n.$$

2.2. Formula for the minimum names

In the same way, $\min_n = C_n + D_n$ (Fig. 1):

$$C_n = 111 \times 10 \uparrow [1 + 2 + \ldots + (n\text{-}2)] + 1011 \times 10 \uparrow [1 + 2 + \ldots + (n\text{-}3)] =$$

$$= (10^2 + 10 + 1) \times 10 \uparrow [(n^2\text{-}3n+2)/2] + (10^3 + 10 + 1) \times 10 \uparrow [(n^2\text{-}5n+6)/2],$$

$$10 \uparrow \{[1 + 2 + \ldots + (n\text{-}4)] + 4\} < D_n < (10 + 1) \times 10 \uparrow \{[1 + 2 + \ldots + (n\text{-}4)] + 3\}.$$

In the decimal form:

$$C_n = 7 \times 2 \uparrow [(n^2\text{-}3n+2)/2] + 11 \times 2 \uparrow [(n^2\text{-}5n+6)/2],$$

$$\underset{\cdot}{\mathbf{D}}_n = 2 \uparrow [(n^2\text{-}7n+20)/2] < D_n < 3 \times 2 \uparrow [(n^2\text{-}7n+18)/2] = \dot{\mathbf{D}}_n.$$

3.3. Some relationships between the maximum and minimum names

The $\min_n$ and $\max_n$ values for n = 4 to 12 and some relationships between them have been calculated (Table 1). The data allow us to express the hypotheses:

$max_{n+1}/max_n \approx 2^n$, where ≈ means 'asymptotic to', $min_{n+1}/max_n \to 7$ as $n \to \infty$. Two other limits are not so obvious.

Table 1. The min_n and max_n values for n = 4 to 12
and some relationships between them.

n	[min_n, max_n]	max_{n+1}/max_n	min_{n+1}/min_n	min_{n+1}/max_n	max_n/min_n
4	[63, 63]	16.22222…	8.04761…	8.04761…	1
5	[507, 1022]	32.04892…	15.61143…	7.74461…	2.01577…
6	[7915, 32754]	64.02008…	30.50953…	7.37262…	4.13821…
7	[241483, 2096914]	128.01075…	62.37541…	7.18322…	8.68348…
8	[15062603, 268427538]	256.00562…	126.32814…	7.08880…	17.82079…
9	[1902830667, 68718960914]	512.00287…	254.37603…	7.04368…	36.11407…
10	[484034528331, 35184305512722]	1024.00145…	510.40210…	7.02166…	72.68966…
11	[247052243600459, 36028779906736402]	2048.00072…	1022.41557…	7.01078…	145.83465…
12	[252590061511541835, 73786967515992695058]	-	-	-	292.12142…
Hypotheses		2^n	?	7	?

4. Proofs of the hypotheses

Owing to rather complicated expressions for B_n and D_n, the main idea of the following calculations is to find the limits by means of lower and upper bounds of the appropriate values.

4.1. $max_{n+1}/max_n \approx 2^n$

To get max_{n+1} we replace n by n+1 in the above formula for max_n:

$$max_{n+1} = 2 \uparrow [(n^2\text{-}3n\text{-}2)/2] \times (2^{2n+1} - 1) + B_{n+1}\text{, where}$$

$$\underset{\cdot}{\mathbf{B}}_{n+1} = 2 \uparrow [(n^2\text{-}5n+2)/2] < B_{n+1} < 2 \uparrow [(n^2\text{-}5n+4)/2] = \dot{\mathbf{B}}_{n+1}.$$

By using the lower and upper bounds for B_n and B_{n+1} we get:

$$(A_{n+1} + \underset{\cdot}{\mathbf{B}}_{n+1}) / (A_n + \dot{\mathbf{B}}_n) < max_{n+1} / max_n < (A_{n+1} + \dot{\mathbf{B}}_{n+1}) / (A_n + \underset{\cdot}{\mathbf{B}}_n).$$

By substituting the above values and passing to the limit we get:

$$2n \lessapprox maxn+1 / maxn \lessapprox 2n.$$

Hence, $max_{n+1} / max_n \approx 2^n$.

4.2. $min_{n+1} / min_n \approx 2^{n-1} + 11/7$

To get min_{n+1} we replace n by n+1 in the above formula for min_n:

$$min_{n+1} = 7 \times 2 \uparrow [(n^2\text{-}n)/2] + 11 \times 2 \uparrow [(n^2\text{-}3n+2)/2] + D_{n+1}\text{, where}$$

$$\underset{\cdot}{\mathbf{D}}_{n+1} = 2 \uparrow [(n^2\text{-}5n+14)/2] < D_{n+1} < 3 \times 2 \uparrow [(n^2\text{-}5n+12)/2] = \dot{\mathbf{D}}_{n+1}.$$

By using the lower and upper bounds for D_n and D_{n+1} we get:

$$(C_{n+1} + \underset{\cdot}{\mathbf{D}}_{n+1}) / (C_n + \dot{\mathbf{D}}_n) < min_{n+1} / min_n < (C_{n+1} + \dot{\mathbf{D}}_{n+1}) / (C_n + \underset{\cdot}{\mathbf{D}}_n).$$

By substituting the above values and passing to the limit we get:

$$2^{n-1} + 11/7 \lessapprox max_{n+1} / max_n \lessapprox 2^{n-1} + 11/7.$$

Hence, $max_{n+1} / max_n \approx 2^{n-1} + 11/7$.

4.3. $\mathbf{min_{n+1} / max_n \to 7}$

In the same way:

$$(C_{n+1} + \underset{\cdot}{\mathbf{D}}_{n+1}) / (A_n + \dot{\mathbf{B}}_n) < min_{n+1} / max_n < (C_{n+1} + \dot{\mathbf{D}}_{n+1}) / (A_n + \underset{\cdot}{\mathbf{B}}_n).$$

By substituting the above values and passing to the limit we get:

$$7 \leq \lim (min_{n+1} / max_n) \leq 7.$$

Hence, $\lim (min_{n+1} / max_n) = 7$.

4.4. $\mathbf{max_n/min_n \approx 2^{n-1}/7}$

In the same way:

$$(A_n + \underset{\cdot}{\mathbf{B}}_n) / (C_n + \dot{\mathbf{D}}_n) < max_n / min_n < (A_n + \dot{\mathbf{B}}_n) / (C_n + \underset{\cdot}{\mathbf{D}}_n).$$

By substituting the above values and passing to the limit we get:

$$2^{n-1} / 7 \lessapprox max_n / min_n \lessapprox 2^{n-1} / 7.$$

Hence, $max_n / min_n \approx 2^{n-1} / 7$.

Interpretation

The tendencies are easy to interpret at a logarithmic scale. The [lg min_n, lg max_n] ranges are getting longer while the gap between them tends to lg 7 = 0.845… (Table 2, Fig. 2).

Table 2. The [lg min_n, lg max_n] ranges for n = 4 to 12.

n	4	5	6	7	8	9	10	11	12
[lg min_n, lg max_n]	[1.80, 1.80]	[2.71, 3.01]	[3.90, 4.52]	[5.38, 6.32]	[7.18, 8.43]	[9.28, 10.84]	[11.68, 13.55]	[14.39, 16.56]	[17.40, 19.87]

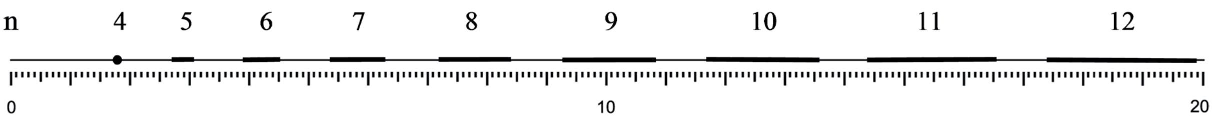

Figure 2. The [lg min_n, lg max_n] ranges for n = 4 to 12 on a real line.

Conclusions

New results are obtained for the combinatorial variety of convex n-hedra (considered as n-acra) previously ordered by their digital names. The ranges [min_n, max_n] rapidly scatter on a real line as $n \to \infty$ in such a regular way that $max_{n+1}/max_n \approx 2^n$ (the 'distance' between the right ends of two nearby ranges), $min_{n+1}/min_n \approx 2^{n-1} + 11/7$ (the 'distance' between the left ends of two nearby ranges), $min_{n+1}/max_n \to 7$ (the 'length' of a gap between two nearby ranges), and $max_n/min_n \approx 2^{n-1}/7$ (the 'length' of a range). The obtained results characterize in detail the strict (without overlapping) ordering of the ranges on a real line.

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References

1. Grünbaum, B. (1967). *Convex Polytopes*. New York: Springer.
2. Voytekhovsky, Y. L. (2016). *Acta Cryst*, A72, 582-585.
3. Voytekhovsky, Y. L. (2017). *Acta Cryst*, A73, 271-273.

ETUDES ON CONVEX POLYHEDRA. 5. TOPOLOGICAL ENTROPIES OF ALL 2907 CONVEX 4- TO 9-VERTEX POLYHEDRA

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Abstract

The topological entropy H_S of all 2907 convex 4- to 9-vertex polyhedra has been calculated from the point of different symmetrical positions of the vertices. It shows a general trend to drop with growing symmetry of polyhedra with many local exceptions. The topological entropy H_V of the same polyhedra has been calculated from the point of different valences of the vertices. It classifies the variety of polyhedra in more detail. The relationships between the H_S and H_V are discussed.

Synopsis

The paper discusses the relationships between the entropies H_S and H_V calculated for all 2907 convex 4- to 9-vertex polyhedra from the point of different symmetrical positions and valences of their vertices, respectively.

Key words

Convex polyhedra, automorphism group orders, symmetry point groups, valences, topological entropy.

1. Introduction

A general theory of convex polyhedra is given in (Grünbaum, 1967). In the series of papers we consider a special problem on the combinatorial variety of convex *n*-hedra rapidly growing with *n*. In Voytekhovsky & Stepenshchikov (2008) and Voytekhovsky (2014) all combinatorial types of convex 4- to 12-hedra and simple (only 3 facets / edges meet at each vertex) 13- to 16-hedra have been enumerated and characterized by automorphism group orders (a.g.o.'s) and symmetry point groups (s.p.g.'s). Asymptotically, almost all *n*-hedra (and *n*-acra, *i.e.* *n*-vertex polyhedra, because of duality) seem to be combinatorially asymmetric (*i.e.* primitive triclinic). A method of naming any convex *n*-acron by a numerical code arising from the adjacency matrix of its edge graph has been suggested in