some with the max. valence), as well as the ordering of the n-acra with the same decompositions of ω depend on the details of their topology.

Acknowledgements

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ETUDES ON CONVEX POLYHEDRA.
3. CONVEX POLYHEDRA WITH MINIMUM AND MAXIMUM NAMES

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Abstract

The paper reports the combinatorial types of convex n-acra (*i.e*. n-vertex polyhedra) for which the minimum (min.) and maximum (max.) names are attained. Hence, min. and max. names can be independently found without generating the whole combinatorial variety of convex n-acra (*ex*., by the routine recurrence Fedorov algorithm) and calculating names for each n-acron.

Synopsis

The combinatorial types of convex n-acra with min. and max. names for any $n \geq 4$ are found. Thus, the latter can be directly calculated from the adjacency matrices of their edge graphs.

Key words

Convex polyhedron, edge graph, adjacency matrix, minimum name, maximum name.

1. Introduction

A general theory of convex polyhedra is given in (Grünbaum, 1967). Here we continue to consider a special problem on the combinatorial variety of convex polyhedra. A hypothesis has been justified in (Voytekhovsky, Stepenshchikov, 2006; Voytekhovsky, 2014) that a fraction of combinatorially asymmetric (*i.e*.

primitive triclinic) convex n-hedra (and n-acra, because of duality) asymptotically tends to 100 % with growing n. It equals 99.238 % for 12-hedra (6336013 of 6384634) and 99.550 % for simple (*i.e.* only 3 facets / edges meet at each vertex) 16-hedra (17411448 of 17490241). Hence, the problem arises: how to discern the overwhelming majority of combinatorially asymmetric convex polyhedra, since the symmetry point groups (s.p.g.'s) and automorphism group orders (a.g.o.'s) do not work?

A method of naming any convex n-acron by a numerical code arising from the adjacency matrix of its edge graph has been suggested in (Voytekhovsky, 2016). Depending on the labeling of the vertices, a number of names of any n-acron equals n! / a.g.o. All of them are strictly connected with each other *via* row and column permutations on the corresponding adjacency matrices. An n-acron can be built using its any name. Classes of convex n-acra are strictly ordered by their names. For example, the ranges [min_n, max_n] of names of 4- to 7-acra are: [63, 63], [507, 1022], [7915, 32754], [241483, 2096914]. For shortness, the above values of min_n and max_n are given in the usual decimal expansion, whereas the names of the polyhedra are initially taken from the adjacency matrices in binary. The relationship between the recurrence Fedorov algorithm to generate the whole combinatorial variety of convex polyhedra (Fedorov, 1893) and the above ordering has been described in (Voytekhovsky, 2017). Below we report the combinatorial types of the convex n-acra with min. and max. names for any n ≥ 4. It allows us to calculate the ranges [min_n, max_n] without using the routine Fedorov algorithm.

2. Convex n-acra with maximum names

All names for all the convex 4- to 7-acra have been calculated in (Voytekhovsky, 2016). The max. names in the classes are resulted from the optimal labelings of the vertices of n-acra looking like "glued tetrahedrons with a common edge" (the latter is marked as 12 in Fig. 1). They are the cyclic 3-polytopes C(n,3) with n vertices (Grünbaum, 1967). Why so?

To prove the statement, let us construct the convex n-acron with a max. name.

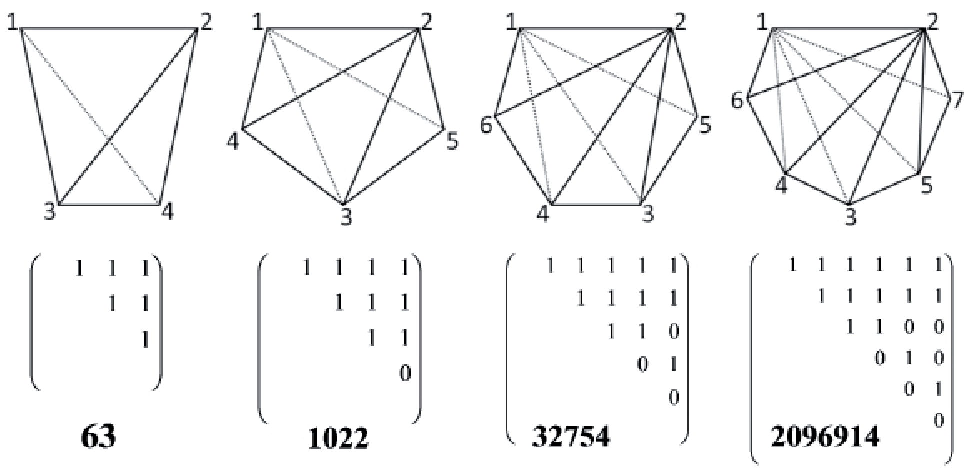

Figure 1. 4- to 7-acra (in axonometric view) with the max. names (in bold) in the classes, corresponding labelings of vertices and adjacency matrices of their edge graphs (upper triangles). They are the cyclic 3-polytopes C(n,3) with n = 4 to 7.

That is, as many 1's as possible should stay in the leftmost position in each line of the upper right triangle of its adjacency matrix. (Step 1) Let us fill in the 1st line of the matrix with 1's. It is ever possible and means that the vertex # 1 is connected with all the others. (Step 2) Let us fill in the 2nd line of the matrix with 1's. It is also ever possible and means that the vertex # 2 is connected with all the others. The resulted configuration looks like (n-2) triangles with the common edge 12. (Step 3) The edge 12 should belong to two facets of an n-acron. Let us take any edge incidental to the vertex # 1 (except 12), as they are equivalent in combinatorial approximation. Obviously, the two form the 1st triangle facet (*ex*., 126 in Fig. 1, right). As we construct a convex n-acron, other vertices should be at one side of this facet. (Step 4) The 2nd facet is taken in the same way. The resulted configuration looks like two triangle facets with (n-2) triangles between them (*ex*., 123, 124 and 125 between 126 and 127 in Fig. 1, right). (Step 5) The unlabeled vertices of the intermediate triangles can be ordered and connected one by one with the edges by rotating the 1st triangle facet towards the 2nd one (*ex*., 126 towards 127 in Fig. 1, right). The resulted configuration looks like a cyclic 3-polytope. In combinatorial approximation, it is uniquely constructed. (Step 6) The optimal labeling of the vertices (except 1 and 2) is as follows. The vertex # 3 should be taken in the middle of a row of unlabeled vertices (for odd n) or in one of two such positions (for even n). Next vertices should be labeled one by one on both sides of # 3, as shown in Fig. 1.

The above procedure can be generalized for any n-acra. Fig. 2 provides the adjacency matrices of 8- to 11-acra corresponding to their max. names.

3. Convex n-acra with minimum names

$$
\begin{pmatrix}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1 \\
 & & & & & & & \\
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\end{pmatrix}
\begin{pmatrix}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 1 & & & & \\
 & & & & & 1 & & & \\
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\begin{pmatrix}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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\end{pmatrix}
\begin{pmatrix}
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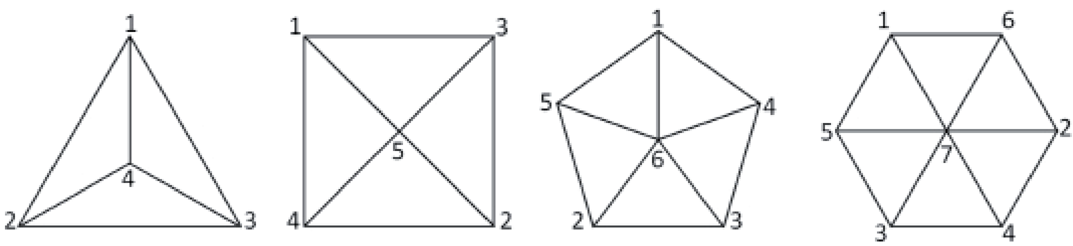
 & & & & & & & & & & 1 \\
 & & & & & & & & & & \\
 & & & & & & & & & &
\end{pmatrix}
$$

Figure 2. The adjacency matrices (upper triangles without 0's) of 8- to 11-acra corresponding to their max. names.

Similarly, it was found in (Voytekhovsky, 2016) that the min. names in the classes of convex 4- to 7-acra are resulted from the optimal labelings of the vertices of pyramids (Fig. 3). (Note that a tetrahedron belongs to both series. That is why its max. and min. names are equal to each other). To prove the statement in a general case, we use the recursive method. How does an 8-acron with the min. name in the class look like? It has a 3-valence vertex of # 1. (Otherwise, any simple 8-acron has less name owing to three 1's at the end of the 1st line of the adjacency matrix.) Let us cut it by the plane passing through its 3 adjacent vertices. The min. name of the obtained convex 7-acron may not be less than that of 7-vertex pyramid. But, if it is bigger than that of a 7-vertex pyramid, we can use it to build the convex 8-acron with a less name than that of an initial 8-acron with the min. name in the class. In result of the contradiction, the 8-acron with the min. name in the class can be attained just from the 7-vertex pyramid by optimal adding of a 3-valence vertex. It is easy to see (by considering some variants), that the best strategy is to add it to the base of the 7-vertex pyramid to produce the 8-vertex one. The above consideration can be successively repeated for n = 9, 10, 11, *etc*. For any n, there is the only n-acron with the min. name in the class.



$$\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ & & 1 & 0 & 1 \\ & & & 0 & 1 \\ & & & & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 0 & 1 & 1 \\ & & 1 & 1 & 0 & 1 \\ & & & 0 & 0 & 1 \\ & & & & 0 & 1 \\ & & & & & 1 \end{pmatrix}$$

63 **507** **7915** **241483**

Figure 3. 4- to 7-acra (in view on a facet) with the min. names (in bold) in the classes, corresponding labelings of vertices and adjacency matrices of their edge graphs (upper triangles).

The optimal labeling of the vertices in a general case is as follows. The apex of any pyramid corresponds to # n. Then the vertex # 1 can be chosen at will. The vertices # n-1 and # n-2 should be taken on both sides of # 1. Other vertices (# 2, 3, *etc*.) should be labeled successively in such a way that the rightmost location of 1's in an adjacency matrix is attained. For example, the adjacency

$$\begin{pmatrix} & & & & 1 & 1 & 1 \\ & & & 1 & & 1 & 1 \\ & & 1 & & 1 & & 1 \\ & & & 1 & & & 1 \\ & & & & & & 1 \\ & & & & & & 1 \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} & & & & & 1 & 1 & 1 \\ & & & & 1 & & 1 & 1 \\ & & & 1 & & 1 & & 1 \\ & & & 1 & 1 & & & 1 \\ & & & & & & & 1 \\ & & & & & & & 1 \\ & & & & & & & 1 \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} & & & & & & 1 & 1 & 1 \\ & & & & & 1 & & 1 & 1 \\ & & & & 1 & & 1 & & 1 \\ & & & 1 & & 1 & & & 1 \\ & & & & 1 & & & & 1 \\ & & & & & & & & 1 \\ & & & & & & & & 1 \\ & & & & & & & & 1 \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} & & & & & & & 1 & 1 & 1 \\ & & & & & & 1 & & 1 & 1 \\ & & & & & 1 & & 1 & & 1 \\ & & & & 1 & & 1 & & & 1 \\ & & & & 1 & 1 & & & & 1 \\ & & & & & & & & & 1 \\ & & & & & & & & & 1 \\ & & & & & & & & & 1 \\ & & & & & & & & & 1 \\ & & & & & & & & & 1 \end{pmatrix}$$

Figure 4. The adjacency matrices (upper triangles without 0's) of 8- to 11-acra corresponding to their min. names.

matrices of 8- to 11-acra corresponding to their min. names are in Fig. 4. (Note that they differ in one 1 for odd and even n).

4. Conclusions

The min. names are attainable for pyramidal convex n-acra with the following combinatorial s.p.g.: -43*m* for n = 4 (tetrahedron), (n-1)*mm* for odd n > 4 and (n-1)*m* for even n > 4. The max. names are attained for convex n-acra of a «glued tetrahedrons» type with the following combinatorial s.p.g.: -43*m* for n = 4 (tetrahedron), -6*m*2 for n = 5 (trigonal bipyramid) and *mm*2 for n > 5.

The above results allow us to directly calculate ranges of names [min_n, max_n] for any n without generating the whole combinatorial variety of convex n-acra (*ex*., by the routine recurrence Fedorov algorithm) and calculating names for all of them. The ranges of names for n = 4 to 12 are as follows: [63, 63], [507, 1022], [7915, 32754], [241483, 2096914], [15062603, 268427538], [1902830667, 68718960914], [484034528331, 35184305512722], [247052243600459, 36028779906736402], [252590061511541835, 73786967515992695058].

All the names of the ranges [max_n + 1, min_{n+1} – 1] (*ex*., [64, 506], [1023, 7914], [32755, 241482], *etc*.) obviously correspond to the adjacency matrices of non-polyhedral graphs. This sufficient but not necessary criterion seems to be new.

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ETUDES ON CONVEX POLYHEDRA. 4. ACCELERATED SCATTERING OF CONVEX POLYHEDRA

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Abstract

The formulas for the minimum (min_n) and maximum (max_n) names in the classes of convex n-acra (*i.e*. n-vertex polyhedra) are found for any n. The asymptotic behavior (as $n \to \infty$) for max_{n+1}/max_n, min_{n+1}/min_n, min_{n+1}/max_n, and max_n/min_n is established. They characterize in detail the accelerated scattering of [min_n, max_n] ranges on a real line.