ETUDES ON CONVEX POLYHEDRA. 2. ORDERING OF CONVEX POLYHEDRA AND THE FEDOROV ALGORITHM

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Abstract

A method of naming any convex polyhedron by a numerical code arising from the adjacency matrix of its edge graph has been suggested earlier. A polyhedron can be built using its name. Classes of convex n-acra (i.e. n-vertex polyhedra) are strictly (without overlapping) ordered by their names. In this paper the relationship between the Fedorov algorithm to generate the whole combinatorial variety of convex polyhedra and the above ordering is described. The convex n-acra are weakly ordered by the maximum extra valencies of their vertices. Thus, non-simple n-acra follow the simple ones for any n.

Synopsis

The relation between the Fedorov algorithm to generate the whole combinatorial variety of convex polyhedra and the ordering of convex n-acra is found. A method to weakly order the convex n-acra by the maximum extra valencies of their vertices is suggested.

Key words

Convex n-acra, the Fedorov algorithm, simple and non-simple polyhedra, extra valencies, weak ordering of n-acra.

1. Introduction

A general theory of convex polyhedra is given in Grünbaum (1967). But some special problems of this field of action are underinvestigated. In the paper (Voytekhovsky, 2014) all combinatorial types of convex 4- to 12-hedra and simple (only 3 facets / edges meet at each vertex) 13- to 16-hedra are enumerated and characterized by automorphism group orders (a.g.o.) and symmetry point groups (s.p.g.). It is found that (a) in a general case, a set of facets and s.p.g. do not fix a polyhedron's type; (b) with growing n, a part of combinatorially asymmetric n-hedra steadily tends to 100 %: 17411448 (99.550 %) for simple 16-hedra (17490241 in total). It seems that, asymptotically, almost all n-hedra (and n-acra, *i.e.* n-vertex polyhedra, because of duality) are combinatorially asymmetric (*i.e.* primitive triclinic). An obvious problem arises: how to manage (name, discern and classify) the overwhelming majority of combinatorially asymmetric convex polyhedral, if the s.p.g. do not work?

The paper (Voytekhovsky, 2016) suggests a method of naming any convex polyhedron (considered as a polyacron) by a numerical code arising from the adjacency matrix of its edge graph. The number of names for any n-acron equals n!/a.g.o. Hence, for any combinatorially asymmetric n-acron it equals n! That is, the combinatorial asymmetry of any polyhedron means its factoriality in the above sense, while the symmetry means its afactoriality. A polyhedron can be uniquely built using any name. It is proved that ranges of names of the n-acra are strictly (without overlapping) ordered with growing n regardless their ordering (by minimum, maximum or any other names) in classes. In this paper the problem of ordering of the n-acra in the classes (for given n) is investigated in terms of the Fedorov algorithm to generate the whole combinatorial variety of convex polyhedra.

2. The Fedorov algorithm

For a given convex n-acron, each labeling of its vertices gives rise to an adjacency matrix of its edge graph. When the rows of the upper right triangle (*excl*. the diagonal with 0's) of this matrix are concatenated and then viewed as a digital sequence representing an integer in binary, a name for a given n-acron results. Various labelings of the vertices can be resulted in different names of the same n-acron. And the question arises, which one should be preferred? As the names are nothing but the integers, we can use minimum (min.), maximum (max.) or some other names for different needs.

It is easy to see that non-simple n-acra follow the simple ones, if ordered by max. names. It is due to any non-simple n-acron having at least 1 vertex of valence 4 or more (*i.e*. extra valence 1 or more). Such a vertex can be labeled as 1 while the adjacent vertices as 2, 3, 4, 5 … *etc*. In this way we get at least four 1's as entries at the beginning of the 1st row of the adjacency matrix and, therefore, of the max. name of any non-simple n-acron. And this max. name is bigger than the max. name of any simple n-acron, which begins with three 1's only. But a non-simple n-acron can include vertices of various extra valencies. Let us consider the situations in details in terms of the Fedorov algorithm.

Fedorov (1893) suggested his recurrence algorithm to generate the whole combinatorial variety of convex polyhedra (Voytekhovsky, 2001). It consists of α, β and γ operations to obtain simple (n+1)-hedra from simple n-hedra ($n \geq 4$) and of ω operation to find non-simple (n+1)-hedra from the previously generated simple (n+1)-hedra. More precisely, α cuts off any vertex with a new triangular facet resulted, β cuts off any edge (2 adjacent vertices) with a new quadrilateral facet resulted and γ cuts off 2 sequential edges (3 vertices) with a new pentagonal facet resulted. The three operations follow the Euler theorem: there are no polyhedra without triangular, quadrilateral and pentagonal facets simultaneously. Finally, ω reduces any edge (*i.e*. joins 2 adjacent vertices), if no triangular facets meet at it (*i.e*. if no facets disappear after this operation).

Given the number F of facets, the number V_s of vertices equals 2F-4 for any convex simple polyhedron. To prove this, we combine the obvious relation $3V = 2E$ and the Euler theorem $F-E+V = 2$, where E is the number of edges. The ω operation reduces V one by one, if repeatedly used for the polyhedra of the same F (Table). Hence, $\omega = V_s-V = 2F-V-4$ is the number of ω operations to get any convex polyhedron from a simple one of the same F: $\omega_{min}=0$ for any simple polyhedron; ω_{max} = F+[F/2]-6, where […] is the integral part of the number. (For shortness, hereinafter we use the letter ω to denote the number of ω operations.) And the problem under investigation is as follows: is it possible to adjust the ordering of the n-acra in the classes by their max. names to that by ω or not?

Table. Numbers of combinatorial types of convex polyhedra with F facets and V vertices.

↓F, V→	4	5	6	7	8	9	10	11	12	13	14	15	16
4	1												
5		1	1										
6		1	2	2	2								
7			2	8	11	8	5						
8			2	11	42	74	76	38	14				
9				8	74	296	633	768	558	219	50		
10				5	76	633	2635	6134	8822	7916	4442	1404	233
11					38	768	6134	25626	64439	104213	112082	79773	36528
12					14	558	8822	64439	268394	709302	1263032	1556952	1338853

Table (continued).

↓F, V→	17	18	19	20	22	24	26	28
11	9714	1249						
12	789749	306470	70454	7595				
13					49566			
14						339722		
15							2406841	
16								17490241

3. Ordering of n-acra

Let us consider the 5- to 7-acra (Figs 1, 2). The 3-gonal bipyramid (max. name 1022, ω=3) follows the 4-gonal pyramid (1011, ω=1) if ordered by these characteristics (1022>1011, 3>1). So, two types of ordering are adjusted for the 5-acra. But the 5-gonal pyramid (32531, ω=2) follows the octahedron (31583, ω=6) and the 6-acron (31582, ω=4). The ranges of the max. names of the 6-acra are overlapped with growing ω as follows: ω=0 [29327], ω=2 [31571, 32531], ω=4 [31582, 32681], ω=6 [31583, 32754]. They are also overlapped for the 7-acra: ω=1 [1984627, 1990799], ω=3 [1990871, 2089235], ω=5 [1993051, 2093699], ω=7 [2057563, 2095881], ω=9 [2057567, 2096914]. Thus, the ordering of the n-acra by the max. names and the ω values are not adjusted in a general case. More careful consideration shows that the situation highly depends on how the total ω value of extra valencies is distributed among the vertices of a non-simple polyacron. For ω=2, two cases are possible: ω=2 (*i.e.* a polyacron has one 3+2=5-valence vertex), and ω=1+1 (*i.e.* two 3+1=4-valence vertices). The two are among the 6-acra. For ω=3, three cases are possible: ω=3 (*i.e.* one 3+3=6-valence vertex), ω=2+1 (*i.e.* one 3+2=5-valence and one 3+1=4-valence vertices), ω=1+1+1 (*i.e.* three 3+1=4-valence vertices). The three are among the 7-acra. All the 5- to

7-acra are characterized by the decompositions of ω into a sum of extra valencies (Figs 1, 2). And the following statement can be formulated.

Statement. For given n, the max. extra valencies of the vertices induce the

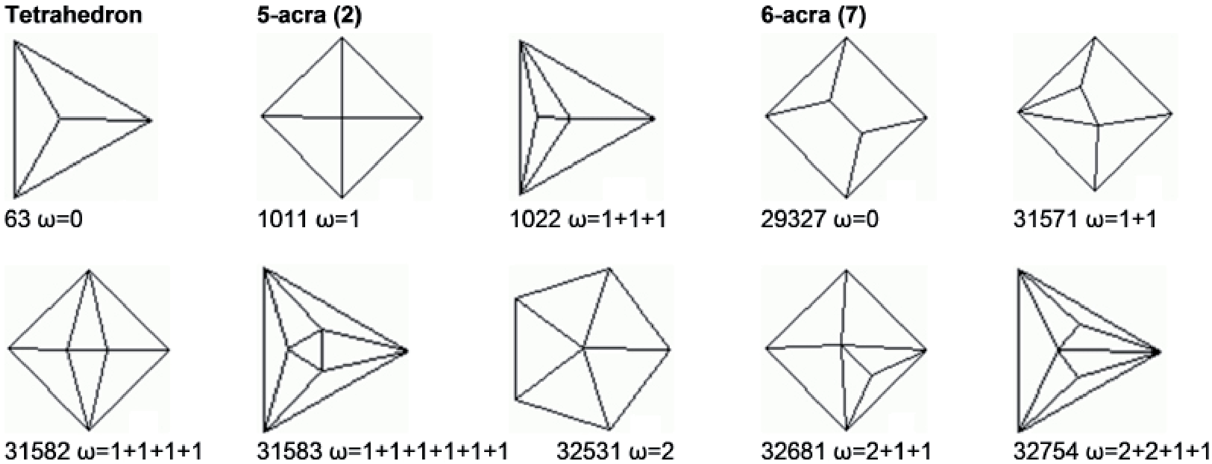

Figure 1. Ordering of 4- to 6-acra by max. names and decompositions of ω into a sum of extra valencies of vertices.

weak order of n-acra corresponded to their strict order by the max. names.

Proof. Let $v_1 \leq v_2 \leq v_3$ be the max. extra valencies taken from the decompositions of ω_1, ω_2, ω_3 of some n-acra. They correspond to the vertices of valencies $3+v_1$, $3+v_2$, $3+v_3$ of these n-acra and can be labeled as 1 while their adjacent vertices as 2, 3, 4, 5 … *etc*. In such a way we guarantee $3+v_1$, $3+v_2$, $3+v_3$ 1's as entries at the beginning of the 1st rows of the adjacency matrices and, therefore, of the max. names of the three n-acra. Hence, their max. names are in the same relation to each other as their max. extra valencies are, and the two orders correspond to each other. There are many n-acra with the same decompositions of ω (Fig. 2). Thus, their ordering by the max. extra valencies induces a weak order.

Corollary. Non-simple n-acra (ω≥1) follow the simple ones (ω=0), when ordered by the max. names. That is due to ω≥1 implying v≥1, while ω=0 implies v=0. But it can be seen in the Table that odd n (*i.e.* V) imply odd ω, while even n imply even ω. Hence, in the latter case non-simple n-acra with ω≥2 follow the simple ones.

4. Conclusions

The relation between the Fedorov algorithm to generate the whole combinatorial variety of convex polyhedra and the ordering of convex n-acra is found. The number of ω operations to get any polyhedron from a simple one with the same number of facets means a lot. For given n, the max. extra valencies of the vertices taken from the decompositions of ω induce a weak order of n-acra corresponded to their strict order by the max. names. Therefore, non-simple n-acra (precisely ω≥2) follow the simple ones (ω=0). Some questions should be investigated. It seems that any decompositions of any ω are represented in the valencies of some n-acra for rather big n. The choice of the vertex # 1 (if there are

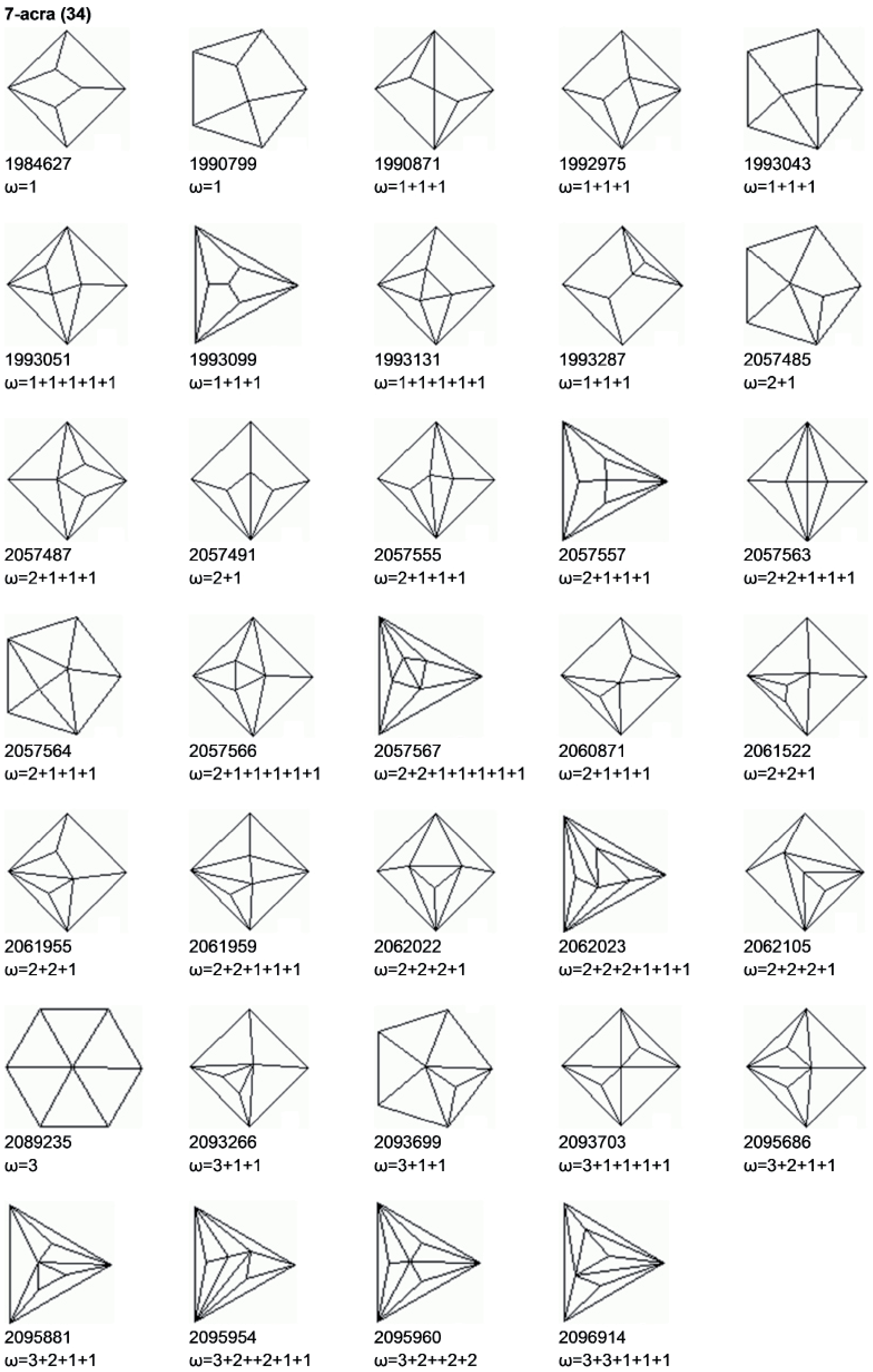

Figure 2. Ordering of 7-acra by max. names and decompositions of ω into a sum of extra valencies of vertices.

some with the max. valence), as well as the ordering of the n-acra with the same decompositions of ω depend on the details of their topology.

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ETUDES ON CONVEX POLYHEDRA.
3. CONVEX POLYHEDRA WITH MINIMUM AND MAXIMUM NAMES

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Abstract

The paper reports the combinatorial types of convex n-acra (*i.e*. n-vertex polyhedra) for which the minimum (min.) and maximum (max.) names are attained. Hence, min. and max. names can be independently found without generating the whole combinatorial variety of convex n-acra (*ex*., by the routine recurrence Fedorov algorithm) and calculating names for each n-acron.

Synopsis

The combinatorial types of convex n-acra with min. and max. names for any $n \geq 4$ are found. Thus, the latter can be directly calculated from the adjacency matrices of their edge graphs.

Key words

Convex polyhedron, edge graph, adjacency matrix, minimum name, maximum name.

1. Introduction

A general theory of convex polyhedra is given in (Grünbaum, 1967). Here we continue to consider a special problem on the combinatorial variety of convex polyhedra. A hypothesis has been justified in (Voytekhovsky, Stepenshchikov, 2006; Voytekhovsky, 2014) that a fraction of combinatorially asymmetric (*i.e*.